Infinite-time Ruin Probability of a Discrete-time Risk Model with Dependent Claims

Rongfei Liu[1, a]

[1]School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731/West Hi-Tech Zone, China

[a]liurongfei078@gmail.com

Keywords: One-sided linear process; Discrete-time risk model; Heavy-tailed innovations; Asymptotic estimate

Abstract. The infinite-time ruin probability of a discrete-time risk model with dependent claims and heavy-tailed innovations is investigated in this paper. The claims are assumed to follow a one-sided linear process with independent and identically distributed (i.i.d.) innovations. Stochastic discount factors, which are independent of the innovations, and constant premium rate are taken into account. As a result, we establish an asymptotic estimate for the infinite-time ruin probability.

Introduction

Consider a discrete-time risk model as follows:

$$U_0 = x, U_n = U_{n-1} q_n^{-1} + c - X_n, n \geq 1. \tag{1.1}$$

x> 0 stands for the initial wealth of an insurer, the constant c > 0 stands for the premium rate and the nonnegative random variable (r.v.) X_n stands for the total claim amount within period n. The investment of the surplus at time n− 1 causes the nonnegative and stochastic discount factor q_n from time n to time n− 1. Thus, U_n is interpreted as the surplus of the insurer at time n. In the terminology of Norberg[1], we call $\{X_n\}_{n\geq 1}$ insurance risks and call $\{q_n\}_{n\geq 1}$ financial risks.

Now we can define the infinite-time ruin probability by

$$\Psi(x) = \mathrm{P}\{\min_{1\leq m<\infty} U_m < 0 \mid U_0 = x\}. \tag{1.2}$$

Many papers discussed the asymptotic behavior of $\Psi(x)$ under the assumption that $\{X_n\}_{n\geq 1}$ and $\{q_n\}_{n\geq 1}$ are two independent sequences of i.i.d. random variables (r.v.s) ([2] and [3], among many others). With the increasing complexity of insurance and reinsurance products, the assumption of independence among $\{X_n\}_{n\geq 1}$ is not enough to depict the real circumstances. Thus, the models of dependent insurance risks are attracting more and more attentions (for examples, [4] ,[5], [6] and [7]).

In the present paper, we suppose that $\{X_n\}_{n\geq 1}$ and $\{q_n\}_{n\geq 1}$ are two independent sequences and $\{q_n\}_{n\geq 1}$ is a i.i.d. sequence. To depict the dependence structure of the claims, we use the following one-sided linear process to describe $\{X_n\}_{n\geq 1}$. Let

$$X_n = \sum\nolimits_{j=1}^{n} j_{n-j} e_j + j_n e_0, n \geq 1, \tag{1.3}$$

where $\{j_n\}_{n\geq 0}$ and e_0 are nonnegative constants with $j_0 > 0$, $\{e_n\}_{n\geq 1}$ is a sequence of i.i.d. and nonnegative r.v.s with common distribution F. Assume that $\{e_n\}_{n\geq 1}$, the innovations of $\{X_n\}_{n\geq 1}$, is independent of $\{q_n\}_{n\geq 1}$. Please see [5] and [8] for more examples of linear processes. We obtain an asymptotic estimate for Ψ(x) when the distribution F belongs to the intersection of the dominated variation class (D) and the long-tailed class (L).

Notations and main result

C represents a positive constant without relation to x and may vary from place to place and all limit relations are for x $\to\infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x)$ **f p** $b(x)$ if $0 < \liminf$ a(x)/b(x) $\le \limsup$ a(x)/b(x) $< \infty$.

By convection, an empty sum is 0 and an empty product is 1. In order to facilitate subsequent expression, we denote

$$q_{n,m} = \prod_{k=n}^{m} q_k, 1 \le n \le m \le \infty; Z_\infty = \sum_{i=1}^{\infty} q_{1,i};$$

$$W_{j,\infty} = \prod_{i=j}^{\infty} q_{1,i} j_{i-j}, 1 \le j < \infty; W_{0,\infty} = \sum_{i=1}^{\infty} q_{1,i} j_i.$$

A distribution F on R has right tail function $\overline{F}$. F belongs to the dominated variation class (D) if

$$\limsup \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty \quad for\ any\ \ 0 < y < 1.$$

F belongs to the long-tailed class (L) if

$$\lim \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1 \quad for\ any\ \ y > 0.$$

Besides that, the upper Matuszewska index J_F^+ and lower Matuszewska index J_F^- (see[9],Ch 2.1.) are used. It is well known that $J_F^+ < \infty$ if F$\in D$. Now, we are ready to state the main result.

Theorem 2.1. Let $\{X_n\}_{n\ge1}$, $\{q_n\}_{n\ge1}$ be mutually independent, $\{X_n\}_{n\ge1}$ be a one-sided linear process introduced in (1.3), and $\{q_n\}_{n\ge1}$ be i.i.d. and nonnegative. If the common distribution function F of the innovations $\{e_j\}_{j\ge1}$, belongs to $D \cap L$, $\sup_{n\ge0} j_n < \infty$, $\sum_{i=1}^{\infty} \mathrm{E} q_1^{pi} < \infty$ for some $J_F^+ < p \le 1$ and $\sum_{j=1}^{\infty} \mathrm{E} W_{j,\infty}^{p_i} < \infty, i = 1,2$ for some $0 < p_1 < J_F^- \le J_F^+ < p_2 < p,$ then it holds that

$$\lim \left| \frac{\Psi(x)}{\sum_{j=1}^{\infty} \mathrm{P}\{e_j W_{j,\infty} > x\}} - 1 \right| = 0. \tag{2.1}$$

Proof of the main result

Some lemmas

By Proposition 2.2.1 in [9], for a distribution F$\in D$, it holds that

$$x^{-p} = o(\overline{F}(x)) \quad for\ any\ \ p > J_F^+. \tag{3.1}$$

From Lemma 3.2 in [10], Lemma 3 in [6] and Lemma 4.1.2 in [3], we have three lemmas.

Lemma 3.1. Let X and Y be two independent and nonnegative random variables, where X is distributed by F. If F$\in D$, then for any fixed δ >0 and $p > J_F^+$, there exists a positive constant C without relation to δ and Y such that for all large x,

$$\mathrm{P}(XY > dx \mid Y) \le C\overline{F}(x)\left[d^{-p} Y^p + 1_{[Y<d]}\right].$$

Lemma 3.2. Let X and Y be two independent and nonnegative random variables, where X is distributed by F. If F$\in D$, then for any fixed δ >0 and $0 < p_1 < J_F^- \le J_F^+ < p_2 < \infty$, there exists a positive constant C without relation to δ and Y such that for all large x,

$$\mathrm{P}(XY > dx \mid Y) \le C\overline{F}(x)\left[d^{-p_1} Y^{p_1} + d^{-p_2} Y^{p_2}\right].$$

Lemma 3.3. Let X and Y be two independent and nonnegative random variables, where X is distributed by F and Y is nondegenerate at 0. If F$\in D \cap L$ and $EY^p < \infty$ for some $p > J_F^+$, then the distribution of XY belongs to $D \cap L$ and $\mathrm{P}(XY > x)$ **f p** $\overline{F}(x)$.

The following lemma will play crucial role in the proof of Theorem 2.1.

Lemma 3.4. Under the conditions of Theorem 2.1, for any $c_0 \geq 0$, it holds that

$$\lim \left| \frac{P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} - c_0 Z_\infty > x\right\}}{\sum_{j=1}^{\infty} P\left\{e_j W_{j,\infty} > x\right\}} - 1 \right| = 0. \tag{3.2}$$

Proof. Under the conditions of Theorem 2.1 and by Cr inequality, we can get

$$EZ_\infty^p = E\left[\left(\sum_{i=1}^{\infty} q_{1,i}\right)^p\right] \leq C\left(\sum_{i=1}^{\infty} Eq_1^{pi}\right) < \infty, \tag{3.3}$$

$$EW_{j,\infty}^p \leq \left(\sup_{n\geq 0} j_n\right)^p E\left[\left(\sum_{i=1}^{\infty} q_{1,i}\right)^p\right] < \infty,\ j \geq 0. \tag{3.4}$$

Firstly, we deal with the upper bound. For any fixed 0 < L < ∞ and k,, we can obtain

$$\begin{aligned} &P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} - c_0 Z_\infty > x\right\} \\ &\leq P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} > x\right\} \\ &\leq P\left\{\bigcup_{j=0}^{\infty} \left\{e_j W_{j,\infty} > x - L\right\}\right\} + P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} > x,\ \bigcap_{j=1}^{k} \left\{e_j W_{j,\infty} \leq x - L\right\}\right\} \\ &:= I_1 + I_2. \end{aligned} \tag{3.5}$$

By Lemma 3.3 and (3.4), the distribution of $e_j W_{j,\infty}$ belongs to $D \cap L$ and

$$P\left\{e_j W_{j,\infty}\right\} \succ \overline{F}(x). \tag{3.6}$$

Then, by Chebyshev's inequality and (3.1), we can get

$$\begin{aligned} I_1 &\leq \sum_{j=1}^{\infty} P\left\{e_j W_{j,\infty} > x - L\right\} + P\left\{e_0 W_{0,\infty} > x - L\right\} \\ &\leq (1 + o(1)) \sum_{j=1}^{\infty} P\left\{e_j W_{j,\infty} > x\right\} + \frac{e_0^p EW_{0,\infty}^p}{(x - L)^p} \\ &\leq (1 + o(1)) \sum_{j=1}^{\infty} P\left\{e_j W_{j,\infty} > x\right\} + o(\overline{F}(x)). \end{aligned} \tag{3.7}$$

By Lemma 3.1, we can obtain that for all large x and any fixed k,

$$\begin{aligned} I_2 &\leq P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} > x,\ \bigcap_{j=1}^{k} \left\{e_j W_{j,\infty} \leq x - L\right\}, \bigcup_{i=1}^{k} \left\{e_i W_{i,\infty} > \frac{x}{k}\right\}\right\} \\ &\leq \sum_{i=1}^{k} P\left\{e_i W_{i,\infty} > \frac{x}{k}, \sum_{j=0, j\neq i}^{\infty} e_j W_{j,\infty} > L\right\} \\ &\leq \sum_{i=1}^{k} C\overline{F}(x) E\left[k^p W_{i,\infty}^p 1_{\{\sum_{j=0, j\neq i}^{\infty} e_j W_{j,\infty} > L\}} + 1_{\{\sum_{j=0, j\neq i}^{\infty} e_j W_{j,\infty} > L\}}\right]. \end{aligned}$$

Then, by (3.4), there exists L* such that for any fixed L ≥ L*,

$$I_2 \leq o(\overline{F}(x)). \tag{3.8}$$

Combining (3.5), (3.7) and (3.8), we can get

$$\begin{aligned} &P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} - c_0 Z_\infty > x\right\} \\ &\leq (1 + o(1)) \sum_{j=1}^{\infty} P\left\{e_j W_{j,\infty} > x\right\} + o(\overline{F}(x)) \\ &\leq (1 + o(1)) \sum_{j=1}^{\infty} P\left\{e_j W_{j,\infty} > x\right\}, \end{aligned}$$

where we used (3.6) in the last step.

Secondly, we deal with the lower bound. For any fixed $0 < D < \infty$ and k, we have

$$
\begin{aligned}
&P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} - c_0 Z_\infty > x\right\} \\
&\geq P\left\{\sum_{j=1}^{k} e_j W_{j,\infty} - c_0 Z_\infty > x\right\} \\
&\geq P\left\{\bigcup_{j=1}^{k} \left\{e_j W_{j,\infty} > x + c_0 Z_\infty\right\}\right\} \\
&\geq \sum_{j=1}^{k} P\left\{e_j W_{j,\infty} > x + c_0 Z_\infty, Z_\infty \leq D\right\} - \sum_{i=1}^{k} \sum_{1\leq j\leq k, j\neq i} P\left\{e_i W_{i,\infty} > x, e_j W_{j,\infty} > x\right\} \\
&\geq \sum_{j=1}^{k} P\left\{e_j W_{j,\infty} > x + c_0 D\right\} - \sum_{j=1}^{k} P\left\{e_j W_{j,\infty} > x + c_0 D, Z_\infty > D\right\} \\
&- \sum_{i=1}^{k} \sum_{1\leq j\leq k, j\neq i} P\left\{e_i W_{i,\infty} > x, e_j W_{j,\infty} > x\right\} \\
&:= L_1 - L_2 - L_3.
\end{aligned} \tag{3.9}
$$

Because the distribution of $e_j W_{j,\infty}$ belongs to $D \cap L$, we can get

$$
L_1 \geq (1 - o(1)) \sum_{j=1}^{k} P\left\{e_j W_{j,\infty} > x\right\}. \tag{3.10}
$$

By Lemma 3.1, we can obtain that for all large x,

$$
L_2 \leq \sum_{j=1}^{k} P\left\{e_j W_{j,\infty} > x, Z_\infty > D\right\} \leq \sum_{j=1}^{k} C\overline{F}(x) \mathrm{E}\left[W_{j,\infty}^{p} 1_{\{Z_\infty > D\}} + 1_{\{Z_\infty > D\}}\right].
$$

Then, by (3.4), there exists D* such that for any fixed D $\geq$ D*,

$$
L_2 \leq o(\overline{F}(x)). \tag{3.11}
$$

By Lemma 3.1 and (3.4), we can get

$$
L_3 \leq \sum_{i=1}^{k} \sum_{1\leq j\leq k, j\neq i} C\overline{F}(x) \mathrm{E}\left[W_{i,\infty}^{p} 1_{\{e_j W_{j,\infty} > x\}} + 1_{\{e_j W_{j,\infty} > x\}}\right] \leq o(\overline{F}(x)). \tag{3.12}
$$

Hence, combining (3.9)-(3.12), we obtain

$$
\begin{aligned}
&P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} - c_0 Z_\infty > x\right\} \\
&\geq (1 - o(1)) \sum_{j=1}^{k} P\left\{e_j W_{j,\infty} > x\right\} - o(\overline{F}(x)) \\
&= (1 - o(1)) \left(\sum_{j=1}^{\infty} - \sum_{j=k+1}^{\infty}\right) P\left\{e_j W_{j,\infty} > x\right\} - o(\overline{F}(x)) \ .
\end{aligned}
$$

By Lemma 3.2 and condition, there exists k* such that for k$\geq$k* and any $0 < p_1 < J_F^- \leq J_F^+ < p_2 < p$,

$$
\sum_{j=k+1}^{\infty} P\left\{e_j W_{j,\infty} > x\right\} \leq C\overline{F}(x) \sum_{j=k+1}^{\infty} \mathrm{E}\left[W_{j,\infty}^{p_1} + W_{j,\infty}^{p_2}\right] \leq o(\overline{F}(x)).
$$

Thus, we can derive

$$
\begin{aligned}
&P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} - c_0 Z_\infty > x\right\} \\
&\geq (1 - o(1)) \sum_{j=1}^{\infty} P\left\{e_j W_{j,\infty} > x\right\} - o(\overline{F}(x)) \\
&\geq (1 - o(1)) \sum_{j=1}^{\infty} P\left\{e_j W_{j,\infty} > x\right\},
\end{aligned}
$$

where we used (3.6) in the last step.

Proof of Theorem 2.1

Proof. From (1.1), we get that for n $\geq$ 1,

$$
U_n = x q_{1,n}^{-1} + \sum_{i=1}^{n} \left[q_{i+1,n}^{-1} (c - X_i)\right]. \tag{3.13}
$$

Substituting (3.13), (1.3) into (1.2) and rewriting the expression, we can derive

$$
\begin{aligned}
\Psi(x) &= P\left\{\min_{0\le m<\infty} U_m q_{1,m} < 0\right\} \\
&= P\left\{\max_{0\le m<\infty} \sum_{i=1}^{m} \left[q_{1,i}(X_i - c)\right] > x\right\} \\
&= P\left\{\max_{0\le m<\infty} \left(\sum_{j=0}^{m} e_j W_{j,m} - cZ_m\right) > x\right\}.
\end{aligned} \tag{3.14}
$$

It is clear that

$$
P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} - cZ_\infty > x\right\} \le \Psi(x) \le P\left\{\sum_{j=0}^{\infty} e_j W_{j,\infty} > x\right\}.
$$

Then, by Lemma 3.4, we can get (2.1).

References

[1]Norberg R. Ruin problems with assets and liabilities of diffusion type. Stochastic Processes and their Applications, 1999; 81(2): 255-269.

[2] Tang Q, Tsitsiashvili G. Randomly weighted sums of subexponential random variables with application to ruin theory. Extremes, 2003; 6: 171-188.

[3] Wang D, Tang Q. Tail probabilities of randomly weighted sums of random variables with dominated variation. Stochastic Models, 2006; 22: 253-272.

[4] Bowers NL, Gerber HU, Hickman JC, Jones DA, Nesbitt CJ.Actuarial Mathematics, 2nd ed.The Society of Actuaries: Schaumburg, IL, 1997.

[5] Gerber HU. Ruin theory in the linear model. Insurance: Mathematics and Economics ,1982; 1: 213-217.

[6] Guo F,Wang D. Uniform asymptotic estimates for ruin probabilities of renewal risk models with exponential Lévy process investment returns and dependent claims. Applied Stochastic Models in Business and Industry, 2013; 29: 295-313.

[7] Peng J, Huang J, Wang D. The ruin probability of a disctete-time risk model with a onesided linear claim process. Communications in Statistics-Theory and Methods, 2011; 40: 4387-4399.

[8] Yang H, Zhang L. Martingale method for ruin probability in an autoregressive model with constant interest rate. Probability in the Engineering and Informational Sciences, 2003; 17: 183-198.

[9] Bingham NH, Goldie CM, Teugels JL. Regular Variation. Cambridge University Press:Cambridge, 1987.

[10] Heyde CC,Wang D. Finite-time ruin probability with an exponential Lévy process investment return and heavy-tailed claims. Advances in Applied Probability, 2009; 41: 206-224.