The Stability Analysis on a Predator-Prey Model with Allee Effect

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Abstract—**This paper mainly focuses on the stability of predator-prey model, in which Allee effect is exerted on preys, people's capture of predators and preys in the system, and the demonstration of positive equilibrium point's existence and stability with the help of Routh-Hurwitz stability discriminance. Combining with numerical modeling, the stability of each population's evolutionary process is also analyzed through choosing Allee effect coefficient and capture coefficient as key parameters, which proves that the two coefficients plays a distinct influence on the change of condition of the system. This paper has particular enlightening significance to the analysis of any condition.**

Keywords-allee effect; predator-prey model; Routh-Hurwitz stability; capture; positive equilibrium point

I. INTRODUCTION

The interaction between predator and prey is the most fundamental intersperse relationship in ecology, which is also considered to be the foundation, on which all biologic chains and the whole ecological system can be established. Lotka-Voltera model is acknowledged as the most classic predator-prey model, but it is unstable[1]. Therefore, the derivation of the interaction between predator and prey is still an important subject of biomathematics [2-4]. There are many researches have been done till now, getting the model more closer to life [5-8].

Allee Effect can be named as Sparse Effect [9]. Allee once pointed out that the so-called Allee Effect referred to the fact that the aggregation of population is beneficial to their growth and existence, but an excessive sparse population will definitely lead to the difficulty of mating. Allee Effect exerts more distinct influence on many endangered species, so it is a hot issue for many[10-11]. Meanwhile, for pursuing economic benefits, people often carry out massive capture of the species resource that is with great economic value in ecological system, which directly leads to the extinction of population seriously.so there exists great practical significance to researches on capture system [12-14].

II. MODELING

The general predator-prey models can be expressed as

$$\begin{cases} \dot{x}(t) = g(x) - \beta xy \\ \dot{y}(t) = e\beta xy - dy \end{cases} \quad (1)$$

Where x represents the population density of the prey at time t, y stands for the population density of the predator at time t, $g(x)$ is the growth rates of the prey, β is the coefficient of the predation, e is conversion rate. All coefficients are positive constants.

Also, a literature give the predator-prey model, in which Allee effect is exerted on preys, the model can look like this

$$\begin{cases} \dot{x}(t) = [r(1 - k^{-1}x) - b(x + a)^{-1}]x - \beta xy \\ \dot{y}(t) = e\beta xy - dy \end{cases} \quad (2)$$

where r is the intrinsic rate of increase of the prey, k represents the environmental capacity. Allee effects can be divided into two types: strong Allee effect and weak Allee effect. The letter b and a represents Allee effect constant, their value range reflects the strength of Allee effects.

Consider applying capture for the predator and prey in the three model. Let m, p represent the prey capture coefficient and predator capture coefficient respectively, these two values are both positive. Then the three model is as follows.

$$\begin{cases} \dot{x}(t) = [r(1 - k^{-1}x) - b(x + a)^{-1}]x - \beta xy - mx \\ \dot{y}(t) = e\beta xy - dy - py \end{cases} \quad (3)$$

In the model(3), let us combined d and p, meanwhile, define $n = p + d$, then we obtain

$$\begin{cases} \dot{x}(t) = [r(1 - k^{-1}x) - b(x + a)^{-1}]x - \beta xy - mx \\ \dot{y}(t) = e\beta xy - ny \end{cases} \quad (4)$$

III. MODEL ANALYSIS

A The existence of positive equilibrium point.

Considering actual meaning of ecological problem, we We will discuss in space $R = \{x/x > 0, y > 0\}$.

Calculating the model (4), we can get four equilibrium point $P_1(0,0)$, $P_4^*(x^*, y^*)$,

$$P_2 = (-km + ar - kr + \sqrt{-4k(b + a(m-r))r + (k(m-r) + ar)^2})/(2r), 0)$$

$$P_3 = (-km + ar - kr - \sqrt{-4k(b + a(m-r))r + (k(m-r) + ar)^2})/(2r), 0)$$

From calculating the model (4), we can get the necessary and sufficient conditions for the existence of the positive equilibrium point $P_3(x^*, y^*)$,

$$b < -(n + ae\beta)(nr + ekmb - ekrb) / (e^2 k \beta^2)$$

In the model(4), the Jacobi matrix at the any equilibrium point is $K = \begin{bmatrix} -m+r+\sigma_1 & -\beta x \\ e\beta y & -n+\sigma_2 \end{bmatrix}$, where $\sigma_2 = e\beta x$,

$$\sigma_1 = -b(x+a)^{-1} + (-2k^{-1}r + b(x+a)^{-2})x - \beta y$$

The characteristic equation of the Jacobi matrix K is

$$\lambda^2 + (\text{tr}K)\lambda + \det K = 0$$

Where $\text{tr}K = m + n - r - \sigma_1 - \sigma_2$,

$\det K = mn - nr - n\sigma_1 - m\sigma_2 + r\sigma_2 + \sigma_1\sigma_2 + e\beta^2 xy$.

Then we get two eigenvalues

$$\lambda_{1,2} = 0.5[(-\text{tr}K) \pm \sqrt{(\text{tr}K)^2 - 4(\det K)}]$$

The stability of positive equilibrium point.

Considering the local stability of equilibrium point at the system (4).Calculating eigenvalue of the Jacobi matrix for each equilibrium point, we can get two eigenvalues, λ_1 and λ_2.Then we will use two different methods to analyzes its stability. First of all, using Routh–Hurwitz theorem we can judge its stability.

For the equilibrium point $P_1(0,0)$, the Jacobi matrix in this system is $K_1 = \begin{bmatrix} r-m-a^{-1}b & 0 \\ 0 & -n \end{bmatrix}$.At this time , its characteristic equations is $\lambda^2 + (m+n+a^{-1}b-r)\lambda + (m+a^{-1}b-r)n = 0$.We can obtain two eigenvalues, $\lambda_1 = -n$ and $\lambda_2 = r - m - a^{-1}b$.Therefore, when $b > a(r-m)$, the system is local stability. But if $b < a(r-m)$, the system is instability.

In order to make the research process convenient, the transpose work is as follows: $t_1 = et$, $y_1 = e^{-1}y$, $r_1 = e^{-1}r$, $b_1 = e^{-1}b$, $m_1 = e^{-1}m$, $n_1 = e^{-1}n$.Substituting t, y, r, b, m, n into t_1, y_1, r_1, b_1, m_1, n_1 separately lead to

$$\begin{cases} \dot{x}(t) = [r(1 - k^{-1}x) - b(x+a)^{-1}]x - \beta xy - mx \\ \dot{y}(t) = \beta xy - ny \end{cases}$$

(5)

For the equilibrium points, $P_2 = ((-km + ar - kr + \sqrt{\theta})/(2r), 0)$, $P_3 = (-(km + ar - kr - \sqrt{\theta})/(2r), 0)$, where $\theta = -4k(b + a(m-r))r + (k(m-r) + ar)^2$.At this time the predator does not exist. In this case, calculating the model (5), we can get that the necessary and sufficient conditions for the existence of P_2 is $b \le [ar + k(r-m)]^2 /(4kr)$, $b \le \min\{ar + k(r-m)]^2 /(4kr), a(r-m)\}$ and $a < k(m-r)/r$ .The necessary and sufficient conditions for the existence of P_3 is $a(r-m) < [ar + k(r-m)]^2 /(4kr)$ and $a < k(m-r)/r$ .We can get

Theorem 1 If the predator does not exist and only the prey exists, the equilibrium point P_2 is instability. If $(kr - km - ar + \sqrt{\theta})/(2r) < \beta^{-1}n$, the equilibrium point P_3 of the system is local stability. If $b = [ar + k(r-m)]^2 /(4kr)$ and $a < r^{-1}k(m-r)$, the equilibrium point P_2 and P_3 degenerate to a point and are high step singular point of the system.

Proof The Jacobi matrix in system (5) is

$$K_2 = \begin{bmatrix} -m+r+\sigma_3 & -\beta x \\ 0 & -n+\sigma_4 \end{bmatrix},$$

Where $\sigma_3 = -b(x+a)^{-1} + (-2k^{-1}r + b(x+a)^{-2})x - \beta y$, $\sigma_4 = \beta x$.

According to the real part of characteristic value of characteristic equation, we can come to the conclusion that the theorem said. If $b = [ar + k(r-m)]^2 /(4kr)$ and $a < r^{-1}k(m-r)$, the equilibrium point P_2 and P_3 degenerate to the point $(0.5r^{-1}k(r-m) - ar, 0)$ .Substituting it into the Jacobi matrix K_2 leads to $|K_2| = 0$ .Therefore it is the high step singular point.

For the equilibrium points $P_4(x^*, y^*)$,the predator and prey are concomitant. In front of this paper we got that the necessary and sufficient conditions for the existence of the positive equilibrium point

$$P_4(x^*, y^*) = (\beta^{-1}n, -k^{-1}\beta^{-2}nr + \beta^{-1}(r-m) - (n+a\beta)^{-1}b) \text{ is}$$

$b < k^{-1}r(k - \beta^{-1}n)(\beta^{-1}n + b)$. Then, we have conclusion as follow

Theorem 2 If the conditions

$b < \min\{k^{-1}r(k - \beta^{-1}n)(\beta^{-1}n + b), rn^2 /(k\beta^2)^{-1}\}$ hold, then the positive equilibrium point P_4 is existent and local stability.

Proof We can get the Jacobi matrix of linear approximate system in the system (5): $K^* = \begin{bmatrix} h'(x) - \beta y - m & -\beta x \\ \beta y & \beta x - n \end{bmatrix}$,

Where $h(x) = [r(1 - k^{-1}x) - (x+a)^{-1}b]x$ .Then according to the Routh-Hurwitz, we have $\det K^*(x^*, y^*) = \beta^2 x^* y^* > 0$,

$$\text{tr}K^*(x^*, y^*) = h'(x^*) - (x^*)^{-1}h(x^*) = -k^{-1}r + (x^*+a)^{-2}b.$$

In order to make the equilibrium point P_4 is local stability, if and only if $\mathrm{tr}K^*(x^*, y^*) < 0$, it said that $b < k^{-1}r(x^* + b)^2$.

Substituting $x^* = \beta^{-1}n$ into it and combining with the necessary and sufficient conditions for the existence of the P, we got that the equilibrium point P_4 is local stability when $b < \min\{k^{-1}r(k - \beta^{-1}n)(\beta^{-1}n + b), rn^2 /(k\beta^2)^{-1}\}$.

Theorem 3 If the positive equilibrium point P is satisfied that $b < k^{-1}r(\beta^{-1}n + b)$ and $m \geq 1$, the system (5) is global asymptotic stability.

Proof Constructing Lyapunov function, we have

$$V(x,y) = x - x^* - x^* \ln((x^*)^{-1}x) + y - y^* - y^* \ln((y^*)^{-1}y),$$

We got $V(x, y) = 0$ around the equilibrium point P_4 and other circumstances are positive. Then $V(x, y)$ is a positive definite function. Then we took the derivative of the system (6).

$$V'(t) = (1 - x^{-1}x^*)(h(x) - \beta xy - mx) + (1 - y^{-1}y^*)(\beta xy - ny)$$
$$= (x - x^*)^2[-k^{-1}r + (x^* + a)^{-1}(x + a)^{-1}b]$$

If $b = 0$, $V'(t) = -k^{-1}r(x - x^*)^2 \leq 0$ always stands up. Thus, the P is global asymptotic stability.

IV. NUMERICAL SIMULATION

To further study the influence that Allee effect and capture have exerted on the system, the effect that the control variables b, m, p have on equation equilibrium state and population quantity will be researched. Here we take parameters: $r = 1$, $k = 8$, $a = 8$, $\beta = 0.8$, $e = 0.8$, $d = 0.5$, the initial value of each variable is as fellows: $x(0) = 6, y(0) = 5$. Substitute each parameter into the systematic equation, and carry out numerical modeling.

If $b = 0$, $m = 0$, $p = 0$, it proves that there exists no Allee effect, and no capture has been done to predators and preys.so the time history graph of prey and predators can be got as follows:
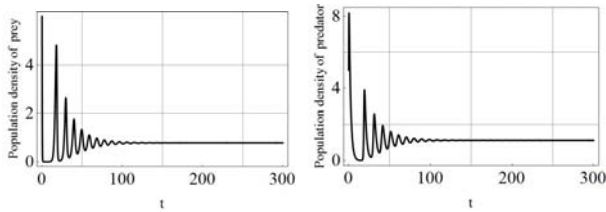

FIGURE I. PHASE DIAGRAM OF THE SYSTEM WITH $b = 0$, $m = 0$, $p = 0$

The graph shows that when there exists no Allee effect and capture, the stable state between preys and predators can be easily reached, so that the system can keep a equilibrium state.

If $b = 1$, $m = 0$, $p = 0$, and there exists Allee effect but no capture, so that the time history graph of preys and predators can be got as follows:(Diagram b)

The graph shows that when there exists Allee effect but no capture has been carried out, preys can reach a stable state in no time although it has been affected, while predators can also reach a stable state but will experience the shortening of population quantity. The Allee effect is very weak, or it will do no good to the existence and growth of population.
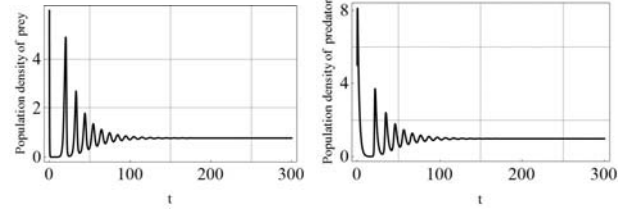

FIGURE II. TIME HISTORY DIAGRAM OF THE SYSTEM WITH $b = 1$, $m = 0$, $p = 0$

If $b = 8$, $m = 0$, $p = 0$, there exists Allee effect but no capture has been carried out. However the Allee effect is strong at the moment, so that the time history graph of preys and predators can be got as follows:
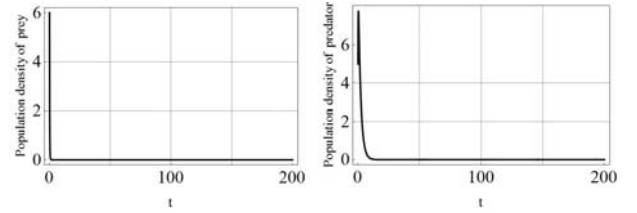

FIGURE III. TIME HISTORY DIAGRAM OF THE SYSTEM WITH $b = 8$, $m = 0$, $p = 0$

The graph shows that the predators and prey all die out due to the strong influence of Allee effect. Allee effect can affect the stability of the system. So when there exists strong Allee effect, the artificial control is needed, so that the extinction of population can be evaded, a stable development can be attained.

The above three situations mainly take the influence that Allee effect have on the system into consideration. The correspondent systematical phase diagram of these three situation can be drew out for a better understanding of the stability of the system. The phase diagram is shown as follows:
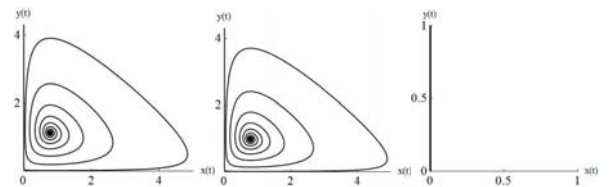

FIGURE IV. PHASE DIAGRAM OF THE SYSTEM WITH $b = 0$, $b = 1$, $b = 8$

The phase diagram also shows that Allee effect can also affect the stability of the system. When Allee effect is weak, species can still keep in stable state. While when Allee effect is strong, it will lead to the extinction of species. Allee effect is a factor that can affect the stability of species.

If $b=1$, $m=0.2$, $p=0.1$, there exists Allee effect and capture is carried out to predators and preys. But if the capture is less, the time history graph of predators and preys can be got as follows:
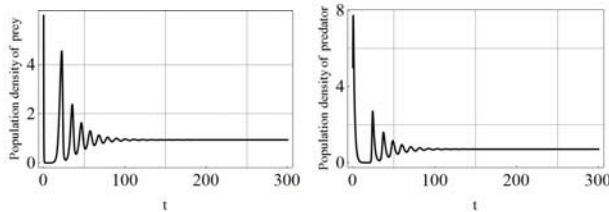

FIGURE V. TIME HISTORY DIAGRAM OF THE SYSTEM WITH $b=1$ $m=0.2$ $p=0.1$

It shows that when a volume of capture is small, it exerts a weak influence on the system, so that the system can easily reach a balanced state, and it produces slight influence on the population quantity of predators and preys, as well as the the growth and development of the population.

If $b=1$, $m=0.9$, $p=0.3$, there exits Allee effect and the capture of predators and preys has been carried out, but the volume of capture is huge.The time history graph of predators and preys can be got as follows:
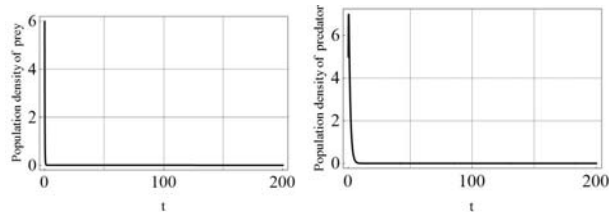

FIGURE VI. TIME HISTORY DIAGRAM OF THE SYSTEM WITH $b=1$ $m=0.9$ $p=0.3$

The graph shows that the predators and preys have died out due to the great influence of huge volume of capture, so that people should take the existence and development of population into account when carrying out captures.

The phase diagrams that are with the same Allee effects but different capture coefficients can also drew from the above two situation also, which are shown as follows:
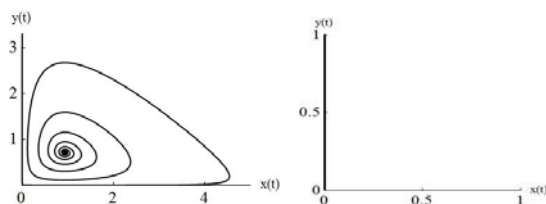

FIGURE VII. PHASE DIAGRAM OF THE SYSTEM WITH $m=0.2$ $p=0.1$; $m=0.9$

Figure 6 and Figure 7 show that the population will become extinct when a huge volume of capture was conducted, which goes against the lasting and stability of the population. But if a small and moderate volume of capture is conducted, the population can still keep a stable state, therefore, capture is also a potential factor that will affect the stability of population

V. CONCLUSIONS

This paper has studied the capture that happened in the two populations of the predator-prey system that is with Allee effect, talked about the stability of the system under several situations. This paper mainly discussed the existence and stability of the positive equilibrium point of the system, and the influences that Allee effect and capture will have on the system by numerical simulation. The capture system is very complicate in nature, and many factors can produce influences on the stability and development of the population. This study demonstrates that Allee effect and capture, as two unstable factors of the capture system, can both affect the stability of the population. No matter the Allee effect is strong or weak, it makes the population die out at last. Under the influence of market economy, although capturing population can gain great value at that time, this value isn't permanent, and the price of which is the extinction of the population. Therefore, people should carry out moderate capture and keep an watchful eye on species for the purpose of getting human beings gain lasting development and evade the high cost, so that the stable state of the whole ecological system can be easily achieved.

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