On the generalized MSSOR method for augmented systems

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Keywords: Augmented systems; MSSOR method; Generalized MSSOR method; Convergence.

Abstract. A generalized MSSOR method (GMSSOR) is presented for solving large sparse augmented linear systems by introducing a new parameter α in MSSOR method, which is an extension of MSSOR method and SOR method. The convergence of GMSSOR method is studied and it is proved that the convergence rate of GMSSOR method is faster than that of MSSOR method by choosing suitable parameter α. A numerical example is given to show that this method is effective.

Introduction

The augmented linear system is taken as

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \tag{1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, and $B \in R^{m\times n}$ is a matrix of full column rank. The augmented system like (1) appears in many different applications of scientific computing, such as the finite element approximation to solve the Navier-Stokes equation [1, 2], the constrained least squares problems and generalized least squares problems [3, 4, 5, 6, 7, 8] and constrained optimization [9].

There are several works for solving the augmented system (1). Santos et al. [5], Santos and Yuan [6] studied preconditioned iterative method for solving the augmented system (1) with $A = I$. Yuan [6, 8] presented preconditioned conjugate gradient methods for solving general systems like (1), where A is symmetric and positive semidefinite and B is rank deficient.

For solving the augmented system (1), Golub et al. [11] presented SOR-like algorithms; Darvishi and Hessari [12] presented SSOR method; Bai [13] presented GSOR method; further, Wu et al.[14] presented MSSOR method. In MSSOR method, although the optimal parameter is employed, the iteration times of MSSOR is larger than those of SOR-like method in [11].

In this paper, we present a generalized MSSOR (GMSSOR) method for the augmented system (1). Also, we discuss its convergence, and it is proved that the convergence rate of GMSSOR method is faster than that of MSSOR method by choosing suitable parameters α. A numerical example is given to show that this method is effective.

The paper is organized as follows. In Section 2, we establish the generalized MSSOR method (GMSSOR). In Section 3, we discuss the convergence region of the method. Finally, we apply GMSSOR method to solve an augmented system.

Algorithm of GMSSOR method

We rewrite the augmented system (1) as

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -q \end{pmatrix},$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, and $B \in R^{m\times n}$ is a matrix of full column rank. To construct GMSSOR method, we consider the follow splitting:

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = D - L - U \,,$$

where

$$D = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix}, \; L = \begin{pmatrix} 0 & 0 \\ B^T & \alpha Q \end{pmatrix}, \; U = \begin{pmatrix} 0 & -B \\ 0 & (1-\alpha)Q \end{pmatrix},$$

and Q is nonsingular symmetric matrix, $0 \le \alpha < 1$.

Let

$$z^{(k)} = \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix}, \; c = \begin{pmatrix} b \\ -q \end{pmatrix}.$$

From SSOR method, we obtain the following scheme:

$$(D - \omega L) z^{(k+\frac{1}{2})} = [(1-\omega)D + \omega U] z^{(k)} + \omega c \,,$$

that is

$$z^{(k+\frac{1}{2})} = \tilde{L}_\omega z^{(k)} + \omega (D - \omega L)^{-1} c \,, \tag{2}$$

where

$$\tilde{L}_\omega = (D - \omega L)^{-1}[(1-\omega)D + \omega U]$$
$$= \begin{pmatrix} (1-\omega)I & -\omega A^{-1}B \\ \dfrac{\omega(1-\omega)}{1-\alpha\omega} Q^{-1}B^T & I - \dfrac{\omega^2}{1-\alpha\omega} Q^{-1}B^T A^{-1} B \end{pmatrix}.$$

Note that

$$D - \omega L = \begin{pmatrix} A & 0 \\ -\omega B^T & (1-\alpha\omega)Q \end{pmatrix}.$$

Since the matrix A is symmetric positive definite and Q is nonsingular, therefore

$$\det(D - \omega L) = (1-\alpha\omega)^2 \det(A)\det(Q) \ne 0$$

if and only if $1 - \alpha\omega \ne 0$, i.e., $\omega \ne \dfrac{1}{\alpha}$.

By the following iterative form, we compute $z^{(k+1)}$ from $z^{(k+\frac{1}{2})}$

$$z^{(k+1)} = \tilde{U}_\omega z^{(k+\frac{1}{2})} + \omega (D - \omega U)^{-1} c \,, \tag{3}$$

where

$$\tilde{U}_\omega = (D - \omega U)^{-1}[(1-\omega)D + \omega L]$$
$$= \begin{pmatrix} (1-\omega)I - \dfrac{\omega^2}{1-(1-\alpha)\omega} A^{-1}BQ^{-1}B^T & -\omega A^{-1}B \\ \dfrac{\omega}{1-(1-\alpha)\omega} Q^{-1}B^T & I \end{pmatrix}.$$

From (2) and (3), we obtain the algorithm of GMSSOR method as follows:

$$z^{(k+1)} = \Omega_\omega z^{(k)} + g \,,$$

where

$$\Omega_\omega = \tilde{U}_\omega \tilde{L}_\omega = \begin{pmatrix} (1-\omega)^2 I - \dfrac{\omega^2(1-\omega)(2-\omega)}{(1-\alpha\omega)[1-(1-\alpha)\omega]} A^{-1}BQ^{-1}B^T & -\omega(2-\omega)A^{-1}B + \dfrac{\omega^3(2-\omega)}{(1-\alpha\omega)[1-(1-\alpha)\omega]} A^{-1}BQ^{-1}B^T A^{-1}B \\ \dfrac{\omega(1-\omega)(2-\omega)}{(1-\alpha\omega)[1-(1-\alpha)\omega]} Q^{-1}B^T & I - \dfrac{\omega^2(2-\omega)}{(1-\alpha\omega)[1-(1-\alpha)\omega]} Q^{-1}B^T A^{-1}B \end{pmatrix}$$

and

$$g=\omega(2-\omega)\begin{pmatrix} A^{-1}b-\dfrac{\omega^2}{(1-\alpha\omega)[1-(1-\alpha)\omega]}A^{-1}BQ^{-1}B^TA^{-1}b+\dfrac{\omega}{(1-\alpha\omega)[1-(1-\alpha)\omega]}A^{-1}BQ^{-1}q \\ \dfrac{\omega}{(1-\alpha\omega)[1-(1-\alpha)\omega]}Q^{-1}B^TA^{-1}b-\dfrac{1}{(1-\alpha\omega)[1-(1-\alpha)\omega]}Q^{-1}q \end{pmatrix}.$$

Given initial vectors $x^{(0)}\in R^n$ and $y^{(0)}\in R^m$, and a relaxation factor $\omega>0$. For $k=0,1,2,\ldots$ until the iteration sequence $\{(x^{(k)},y^{(k)})^T\}$ is convergent, compute

$$\begin{cases} y^{(k+1)}=y^{(k)}+\dfrac{\omega(2-\omega)}{(1-\alpha\omega)[1-(1-\alpha)\omega]}Q^{-1}B^T[(1-\omega)x^{(k)}-\omega A^{-1}By^{(k)}+\omega A^{-1}b]-\dfrac{\omega(2-\omega)}{(1-\alpha\omega)[1-(1-\alpha)\omega]}Q^{-1}q \\ x^{(k+1)}=(1-\omega)^2x^{(k)}-\omega A^{-1}B[y^{(k+1)}+(1-\omega)y^{(k)}]+\omega(2-\omega)A^{-1}b \end{cases},$$

where Q is an approximate (preconditioning) matrix of the Schur complement matrix $B^TA^{-1}B$.

For the special case, the GMSSOR method is the same as the method studied in Darvishi and Hessari [12] with $\alpha=0$, and in Wu et al.[14] with $\alpha=\dfrac{1}{2}$.

Convergence analysis

Theorem 1. Let λ ($\lambda\neq(1-\omega)^2$ and $\lambda\neq1$) be an eigenvalue of Ω_ω. Then there exists a nonzero real eigenvalue μ of $Q^{-1}B^TA^{-1}B$, such that the following functional relationship holds

$$(1-\lambda)[\lambda-(1-\omega)^2]=\frac{\lambda\omega^2(2-\omega)^2}{(1-\alpha\omega)[1-(1-\alpha)\omega]}\mu. \tag{4}$$

Proof. Evidently, the eigenvalue μ of $Q^{-1}B^TA^{-1}B$ are real and nonzero if Q is a nonsingular symmetric positive definite matrix. Let λ ($\lambda\neq(1-\omega)^2$ and $\lambda\neq1$) be a nonzero eigenvalue of Ω_ω with eigenvector $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}$, that is $\Omega_\omega x=\lambda x$. By calculation, we obtain that

$$\begin{pmatrix}(1-\omega)^2I & -\omega(1-\omega)A^{-1}B \\ \omega(1-\omega)Q^{-1}B^T & (1-\alpha\omega)[1-(1-\alpha)\omega]I-\omega^2Q^{-1}B^TA^{-1}B\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$
$$=\lambda\begin{pmatrix}I & \omega A^{-1}B \\ \omega Q^{-1}B^T & (1-\alpha\omega)[1-(1-\alpha)\omega]I-\omega^2Q^{-1}B^TA^{-1}B\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix},$$

that is

$$\begin{cases}[(1-\omega)^2-\lambda]x_1=\omega(1+\lambda-\omega)A^{-1}Bx_2 \\ (1-\lambda)(1-\alpha\omega)[1-(1-\alpha)\omega]x_2-(1-\lambda)\omega^2Q^{-1}B^TA^{-1}Bx_2=\omega(\omega-1-\lambda)Q^{-1}B^Tx_1\end{cases}$$

From the first equation, we get

$$x_1=\frac{\omega(1+\lambda-\omega)}{(1-\omega)^2-\lambda}A^{-1}Bx_2,$$

which means $x_2\neq0$ (if $x_2=0$, we get $x=0$, which contradicts to that x is a eigenvector of Ω_ω). Taking the place of x_1 in the second equation yields

$$(1-\lambda)(1-\alpha\omega)[1-(1-\alpha)\omega]x_2-(1-\lambda)\omega^2Q^{-1}B^TA^{-1}Bx_2=\omega(\omega-1-\lambda)\frac{\omega(1+\lambda-\omega)}{(1-\omega)^2-\lambda}Q^{-1}B^TA^{-1}Bx_2.$$

By simple manipulations, it is easy to get that

$$(1-\lambda)[\lambda-(1-\omega)^2]x_2=\frac{\lambda\omega^2(2-\omega)^2}{(1-\alpha\omega)[1-(1-\alpha)\omega]}Q^{-1}B^TA^{-1}Bx_2.$$

i.e.,

$$Q^{-1}B^TA^{-1}Bx_2=\frac{(1-\lambda)[\lambda-(1-\omega)^2](1-\alpha\omega)[1-(1-\alpha)\omega]}{\lambda\omega^2(2-\omega)^2}x_2.$$

Let

$$\mu=\frac{(1-\lambda)[\lambda-(1-\omega)^2](1-\alpha\omega)[1-(1-\alpha)\omega]}{\lambda\omega^2(2-\omega)^2}.$$

Then $Q^{-1}B^TA^{-1}Bx_2=\mu x_2$. From $x_2\neq 0$, we obtain that μ is a eigenvalue of $Q^{-1}B^TA^{-1}B$ and

$$(1-\lambda)[\lambda-(1-\omega)^2]=\frac{\lambda\omega^2(2-\omega)^2}{(1-\alpha\omega)[1-(1-\alpha)\omega]}\mu. \qquad ■$$

Remark 1. The case $\alpha=0$ is Theorem 1 of the [12], and the case $\alpha=\frac{1}{2}$ is Theorem 2 of the [14].

Lemma 1 ([15]). Consider the quadratic equation $x^2-bx+c=0$, where b and c are real numbers. Both roots of the equation are less than one in modulus if and only if $|c|<1$ and $|b|<1+c$.

Theorem 2. Let $A\in R^{m\times m}$ be a symmetric positive definite matrix, and $B\in R^{m\times n}$ be a matrix of full column rank. Assume that all eigenvalues μ of $Q^{-1}B^TA^{-1}B$ are real. Then we have the following cases for relaxation parameter of GMSSOR method:

Case 1: If $\alpha=0$, we have

(1) if $0<\mu<4(\sqrt{2}-1)$, then GMSSOR method converges for all $0<\omega<1$;

(2) if $\mu\geq 4(\sqrt{2}-1)$, then GMSSOR method converges for all ω such that

$$0<\omega\leq\frac{4-\rho-\sqrt{\rho^2+8\rho-16}}{4},$$

where ρ is the spectral radius of $Q^{-1}B^TA^{-1}B$.

Case 2: If $\alpha=\frac{1}{2}$, we have

(1) if $0<\mu\leq\frac{1}{4}$, then GMSSOR method converges for all $0<\omega<2$;

(2) if $\mu=\frac{1}{2}$, then GMSSOR method converges for all $0<\omega<1$;

(3) if $\frac{1}{4}<\mu$ and $\mu\neq\frac{1}{2}$, then GMSSOR method converges for all $0<\omega<\frac{1-\sqrt{4\rho-1}}{1-2\rho}<2$, where ρ is the spectral radius of $Q^{-1}B^TA^{-1}B$.

Case 3: If $0<\alpha<\frac{1}{2}$, we have

(1) if $\mu=2$, then GMSSOR method converges for all $0<\omega<1-\frac{\sqrt{3}}{3}$ or $1+\frac{\sqrt{3}}{3}<\omega<\frac{1}{1-\alpha}$;

(2) if $0<\mu<2$, then GMSSOR method converges for all $0<\omega<1-\sqrt{\frac{-(1+\rho)+\sqrt{1+4\rho}}{2-\rho}}$ or $1+\sqrt{\frac{-(1+\rho)+\sqrt{1+4\rho}}{2-\rho}}<\omega<\frac{1}{1-\alpha}$;

(3) if $\mu>2$, then GMSSOR method converges for all

$$\max\{0,1-\sqrt{\frac{-(1+\rho)-\sqrt{1+4\rho}}{2-\rho}}\}<\omega<\min\{\frac{1}{1-\alpha},1+\sqrt{\frac{-(1+\rho)-\sqrt{1+4\rho}}{2-\rho}}\}$$

or

$$\max\{0,1-\sqrt{\frac{-(1+\rho)+\sqrt{1+4\rho}}{2-\rho}}\}<\omega<\min\{\frac{1}{1-\alpha},1+\sqrt{\frac{-(1+\rho)+\sqrt{1+4\rho}}{2-\rho}}\},$$

where ρ is the spectral radius of $Q^{-1}B^TA^{-1}B$.

Case 4: If $\frac{1}{2}<\alpha<1$, we have

(1) if $\mu=2$, then GMSSOR method converges for all $0<\omega<1-\frac{\sqrt{3}}{3}$ or $1+\frac{\sqrt{3}}{3}<\omega<\frac{1}{\alpha}$;

(2) if $0<\mu<2$, then GMSSOR method converges for all

$$0<\omega<1-\sqrt{\frac{-(1+\rho)+\sqrt{1+4\rho}}{2-\rho}} \text{ or } 1+\sqrt{\frac{-(1+\rho)+\sqrt{1+4\rho}}{2-\rho}}<\omega<\frac{1}{\alpha};$$

(3) if $\mu>2$, then GMSSOR method converges for all

$$\max\{0,1-\sqrt{\frac{-(1+\rho)-\sqrt{1+4\rho}}{2-\rho}}\}<\omega<\min\{\frac{1}{\alpha},1+\sqrt{\frac{-(1+\rho)-\sqrt{1+4\rho}}{2-\rho}}\}$$

or

$$\max\{0,1-\sqrt{\frac{-(1+\rho)+\sqrt{1+4\rho}}{2-\rho}}\}<\omega<\min\{\frac{1}{\alpha},1+\sqrt{\frac{-(1+\rho)+\sqrt{1+4\rho}}{2-\rho}}\},$$

where ρ is the spectral radius of $Q^{-1}B^TA^{-1}B$.

Proof. Let λ ($\lambda\neq(1-\omega)^2$ and $\lambda\neq1$) be an eigenvalue of Ω_ω. From Theorem 1, there exists a nonzero real eigenvalue μ of $Q^{-1}B^TA^{-1}B$, such that Equation (4) holds. After some simple manipulations on Equation (4), we get

$$\lambda^2-[1+(\omega-1)^2-\frac{\omega^2(2-\omega)^2}{(1-\alpha\omega)[1-(1-\alpha)\omega]}\mu]\lambda+(\omega-1)^2=0.$$

From Lemma 1, $|\lambda|<1$ if and only if $|(\omega-1)^2|<1$ and

$|1+(1-\omega)^2-\frac{\omega^2(2-\omega)^2}{(1-\alpha\omega)[1-(1-\alpha)\omega]}\mu|<1+(\omega-1)^2$. Also, the above inequalities are equal to

$$|\omega-1|<1,$$

$$-1-(\omega-1)^2<1+(\omega-1)^2-\frac{\omega^2(2-\omega)^2}{(1-\alpha\omega)[1-(1-\alpha)\omega]}\mu<1+(\omega-1)^2.$$

Hence, it is easy to get that

$$0<\omega<2,$$

$$-\frac{\omega^2(2-\omega)^2}{(1-\alpha\omega)[1-(1-\alpha)\omega]}\mu<0, \quad (5)$$

and

$$2+2(\omega-1)^2-\frac{\omega^2(2-\omega)^2}{(1-\alpha\omega)[1-(1-\alpha)\omega]}\mu>0. \quad (6)$$

It is easy to see that inequality (5) is true if $\mu>0$, $\omega\neq0,2$ and

$$(1-\alpha\omega)[1-(1-\alpha)\omega]>0. \quad (7)$$

Solving the inequality $(1-\alpha\omega)[1-(1-\alpha)\omega]>0$, and from $0<\omega<2$, we obtain the following Cases:

Case 1: if $\alpha=0$, we have $0<\omega<1$;

Case 2: if $\alpha=\frac{1}{2}$, we have $0<\omega<2$;

Case 3: if $0<\alpha<\frac{1}{2}$, we have $0<\omega<\frac{1}{1-\alpha}$;

Case 4: if $\frac{1}{2}<\alpha<1$, we have $0<\omega<\frac{1}{\alpha}$.

From (6), we have

Case 1: if $\alpha=0$, we have

$$2+2(\omega-1)^2-\frac{\omega^2(2-\omega)^2}{1-\omega}\mu>0,$$

which is Theorem 2 of [12], so we obtain results of the Case 1.

Case 2: if $\alpha=\frac{1}{2}$, we have

$$2+2(\omega-1)^2-4\omega^2\mu>0,$$

which is Theorem 3 of [14], so we obtain results of the Case 2.

Case 3: if $0<\alpha<\frac{1}{2}$, i.e. $0<\omega<\frac{1}{1-\alpha}$, we have

$$2+2(\omega-1)^2-\frac{\omega^2(2-\omega)^2}{1-\omega}\mu>2+2(\omega-1)^2-[\frac{1}{(\omega-1)^2}-2+(\omega-1)^2]\mu.$$

So we obtain condition(s) such that

$$2+2(\omega-1)^2-[\frac{1}{(\omega-1)^2}-2+(\omega-1)^2]\mu>0,$$

That is

$$(2-\mu)(\omega-1)^4+(2+2\mu)(\omega-1)^2-\mu>0. \qquad (8)$$

Let $T=(\omega-1)^2>0$, then (8) is changed into

$$(2-\mu)T^2+(2+2\mu)T-\mu>0. \qquad (9)$$

For (9), we have following cases

(1) If $\mu=2$, solving the inequality (9), and from $0<\omega<\frac{1}{1-\alpha}$, we have $0<\omega<1-\frac{\sqrt{3}}{3}$ or $1+\frac{\sqrt{3}}{3}<\omega<\frac{1}{1-\alpha}<2$.

(2) If $0<\mu<2$, from (9) we have $\Delta=4(1+4\mu)>0$, solving the inequality (9), we have

$$0<\omega<1-\sqrt{\frac{-(1+\mu)+\sqrt{1+4\mu}}{2-\mu}} \text{ or } 1+\sqrt{\frac{-(1+\mu)+\sqrt{1+4\mu}}{2-\mu}}<\omega<\frac{1}{1-\alpha}<2.$$

As for any μ we have $\mu\le\rho$, where ρ is the spectral radius of $Q^{-1}B^TA^{-1}B$, since

$f(\mu)=1-\sqrt{\frac{-(1+\mu)+\sqrt{1+4\mu}}{2-\mu}}$ is the monotone decreasing function in $0<\mu<2$, and

$g(\mu)=1+\sqrt{\frac{-(1+\mu)+\sqrt{1+4\mu}}{2-\mu}}$ is the monotone increasing function in $0<\mu<2$, hence Case 3 (2) is true.

(3) If $\mu>2$, from (9), we have $\Delta=4(1+4\mu)>0$, solving the inequality (9), and from $0<\omega<\frac{1}{1-\alpha}$, we have

$$\max\{0,1-\sqrt{\frac{-(1+\mu)+\sqrt{1+4\mu}}{2-\mu}}\}<\omega<\min\{\frac{1}{1-\alpha},1+\sqrt{\frac{-(1+\mu)+\sqrt{1+4\mu}}{2-\mu}}\},$$

or

$$\max\{0,1-\sqrt{\frac{-(1+\mu)-\sqrt{1+4\mu}}{2-\mu}}\}<\omega<\min\{\frac{1}{1-\alpha},1+\sqrt{\frac{-(1+\mu)-\sqrt{1+4\mu}}{2-\mu}}\}\ .$$

As for any μ we have $\mu\le\rho$, where ρ is the spectral radius of $Q^{-1}B^TA^{-1}B$, since $f(\mu)=1-\sqrt{\frac{-(1+\mu)+\sqrt{1+4\mu}}{2-\mu}}$ and $h(\mu)=1+\sqrt{\frac{-(1+\mu)-\sqrt{1+4\mu}}{2-\mu}}$ are the monotone decreasing function in $\mu>2$, $g(\mu)=1+\sqrt{\frac{-(1+\mu)+\sqrt{1+4\mu}}{2-\mu}}$ and $w(\mu)=1-\sqrt{\frac{-(1+\mu)-\sqrt{1+4\mu}}{2-\mu}}$ is the monotone increasing function in $\mu>2$, hence We should have

$$\max\{0,1-\sqrt{\frac{-(1+\rho)-\sqrt{1+4\rho}}{2-\rho}}\}<\omega<\min\{\frac{1}{1-\alpha},1+\sqrt{\frac{-(1+\rho)-\sqrt{1+4\rho}}{2-\rho}}\}$$

or

$$\max\{0,1-\sqrt{\frac{-(1+\rho)+\sqrt{1+4\rho}}{2-\rho}}\}<\omega<\min\{\frac{1}{1-\alpha},1+\sqrt{\frac{-(1+\rho)+\sqrt{1+4\rho}}{2-\rho}}\},$$

where ρ is the spectral radius of $Q^{-1}B^TA^{-1}B$, thus the Case 3 is true.

Case 4. if $\frac{1}{2}<\alpha<1$, Similar to the proof of Case 3, it is easy to get the Case 4, this completes the proof. ■

Numerical example

Example. Let the augmented system (1) in which

$$A=\begin{pmatrix} I\otimes T+T\otimes I & 0 \\ 0 & I\otimes T+T\otimes I \end{pmatrix}\in R^{2p^2\times 2p^2}.$$

$$B=\begin{pmatrix} I\otimes F \\ F\otimes I \end{pmatrix}\in R^{2p^2\times p^2},$$

and

$$T=\frac{1}{h^2}\cdot tridiag(-1,2,-1)\in R^{p\times p}\ ,F=\frac{1}{h}\cdot tridiag(-1,1,0)\in R^{p\times p},$$

with $\otimes$ being the Kronecker product symbol and $h=\frac{1}{p+1}$, $S=tridiag(a,b,c)$ is a tridiagonal matrix with $S_{i,i}=b$, $S_{i-1,i}=a$, and $S_{i,i+1}=c$ for appropriate i.

For this example, we set $m=2p^2$ and $n=p^2$. Hence, the total number of variable is $m+n=3p^2$. We choose the matrix $Q=\frac{2}{3}B^TA^{-1}B$. In our computations, the initial vector is $(x^{(0)^T},y^{(0)^T})=0$, and terminated if the current iteration satisfy $ERR<10^{-9}$, where "ERR" is the norm of absolute value of error vector defined by

$$ERR=\frac{\sqrt{\|x^{(k)}-x^*\|_2^2+\|y^{(k)}-y^*\|_2^2}}{\sqrt{\|x^{(0)}-x^*\|_2^2+\|y^{(0)}-y^*\|_2^2}},$$

and "RES" is defined to be

$$RES=\sqrt{\|b-Ax^{(k)}-By^{(k)}\|_2^2+\|q^{(k)}-B^Tx^{(k)}\|_2^2},$$

where $(x^{(k)},y^{(k)})^T$ is the final approximate solution (see Table1).

We chose the right hand side vector $(b^T, q^T)^T \in R^{m+n}$ such that the exact solution of the augmented system (1) is $((x^*)^T, (y^*)^T)^T = (1,1,\ldots 1) \in R^{m+n}$. The numerical example shows that GMSSOR method is feasible and effective. The determination of optimum value of the parameter α needs further studies.

Table 1: IT (iterative times) and RES for the example

p	Algorithm	α	ω	$\rho(\Omega_\omega)$	IT	RES
$p=8$	MSSOR	$\alpha=\frac{1}{2}$	$\omega=0.1$	0.9769	212	1.0523e-007
	GMSSOR	$\alpha=\frac{1}{4}$	$\omega=0.1$	0.9757	199	4.5750e-007
$p=8$	MSSOR	$\alpha=\frac{1}{2}$	$\omega=0.2$	0.9408	105	1.0523e-007
	GMSSOR	$\alpha=\frac{1}{4}$	$\omega=0.2$	0.9324	93	2.6089e-007

Acknowledgements

This Project Supported by Natural Science Basic Research Plan in Shaanxi Province of China (No.2013JM1001) and the key project of Baoji University of Arts and Sciences (No.ZK16050).

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