The Ulam stability of Jensen-Quartic functional equation

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Abstract. This article gives the definition of the Jensen-Quartic functional equation, obtain the general solution of the Jensen-Quartic functional equation. Research the realations between Jensen-Quartic and Cauchy-Quartic functional equation, and finally proved the Ulam stability of Jensen-Quartic on Banach space.

1.Introduction

The stability problem of functional equations originated from a question of Ulam[1] concern the stability of group homomorphisms, Hyers[2]gave a affirmative answer to the question of Ulam for Banach spaces. In 1978, Rassias[3] has proved lot of influence in the development of Ulam stability of functional equation. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors([4],[5]).

Through this section, we assume that X , Y are real vector spaces.

The function $f: X \to Y$ is called Cauchy functiona if and only if it satisfied the functional equation $f(x+y) = f(x) + f(y)$; If the function f satisfied $2f(\frac{x+y}{2}) = f(x) + f(y)$, then we called it Jensen functiona, It is obviously that a Cauchy function must be a Jensen function.

Functional equation $f(2x+y)+f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y)$, (1.1)

is called quartic functional equation.

Quartic functional equation has very close relationship with Jordan-von Neumann equation, it often used to characterize inner product spaces. Lee and Sung[6]established the general solution of quartic functional equation.

Lemma 1.1 A function $f: X \to Y$ satisfies the functional equation (1.1) if and only if there exists a

symmetric biquadratic function $B: X^2 \to Y$ such that $f(x) = B(x,x)$ for all $x \in X$, in fact

$$B(x,y) = \frac{1}{12}[f(x+y) + f(x-y) - 2f(x) - 2f(y)]. \qquad (1.2)$$

Recently people begin to research Ulam stability of multi-variables functional equation, such as Chu, Ku and Park[7]researched the Ulam stability of partial derivations of n-variable functions.

In this paper, based on the above research, we defined the Jensen-Quartic functional equation, and by means of the research of quartic functional equation. Obtained the general solution of

Jensen-Quartic functional equation, and proved the Ulam stability on Banach spaces.

Definition 1.2 A function $f: X^2 \to Y$ is called Cauchy-Quartic function if for $\forall x_1, x_2, y_1, y_2 \in X$, it satisfied the Cauchy-Quartic functional equation:

$$f(x_1+x_2, 2y_1+y_2)+f(x_1+x_2, 2y_1-y_2)=4f(x_1, y_1+y_2)+4f(x_1, y_1-y_2)+24f(x_1, y_1)-6f(x_1, y_2)+$$
$$4f(x_2, y_1+y_2)+4f(x_2, y_1-y_2)+24f(x_2, y_1)-6f(x_2, y_2). \qquad (1.3)$$

Definition 1.3 A function $f: X^2 \to Y$ is called Jensen-Quartic function if for $\forall x_1, x_2, y_1, y_2 \in X$, it satisfied the Jensen-Quartic functional equation:

$$f(\frac{x_1+x_2}{2}, 2y_1+y_2)+f(\frac{x_1+x_2}{2}, 2y_1-y_2)=2f(x_1, y_1+y_2)+2f(x_1, y_1-y_2)+12f(x_1, y_1)-3f(x_1, y_2)+$$
$$2f(x_2, y_1+y_2)+2f(x_2, y_1-y_2)+12f(x_2, y_1)-3f(x_2, y_2). \qquad (1.4)$$

2. The general solution of equation (1.4)

Lemma 2.1 A function $f: X^2 \to Y$ satisfies the functional equation (1.4) if and only if f is Jensen for the first variable, and is quartic for the second variable. i.e. for $\forall x_1, x_2, y_1, y_2 \in X$, we have

$$2f(\frac{x_1+x_2}{2}, y)=f(x_1, y)+f(x_2, y)\ f(x, 2y_1+y_2)+f(x, 2y_1-y_2)=4f(x, y_1+y_2)+4f(x, y_1-y_2)$$
$$+24f(x, y_1)-6f(x, y_2).$$

Proof: The necessity is clear, we only need to proof the sufficiency. First, we proof f is quartic concerned the second variable. In (1.4), let $x_1 = x_2 = x$, then we get

$$f(x, 2y_1+y_2)+f(x, 2y_1-y_2)=4f(x, y_1+y_2)+4f(x, y_1-y_2)+24f(x, y_1)-6f(x, y_2).$$

It is clear that f is quartic for the second variable, now we proof f is Jensen for the first variable.

In (1.4), let $x_1 = x_2 = x, y_1 = y_2 = 0$, we obtain $f(x,0)=0$; In (1.4), let $y_2 = 0$, thus we have $f(\frac{x_1+x_2}{2}, 2y_1)=8f(x_1, y_1)+8f(x_2, y_1)$; if we put $x_1 = x_2 = x$, $y_1 = y$, then we have $f(x, 2y)=16f(x, y)$, thus $2f(\frac{x_1+x_2}{2}, 2y_1)=32f(\frac{x_1+x_2}{2}, y_1)$, from which we have $2f(\frac{x_1+x_2}{2}, y_1)=f(x_1, y_1)$ $+f(x_2, y_1)$. The following theorem is an immediate consequence of lemma 2.1.

Theorem 2.2 A function $f: X^2 \to Y$ satisfies the functional equation (1.4) if and only if there exist a function $A: X^3 \to Y$, and a function $B: X^3 \to Y$ such that $f(x,y) = A(x,y,y) + B(y,y)$ for all $x, y \in X$, and function A is addictive for the first variable, symmetric and quartic for the last two variables; function B is symmetric for each fixed variable and is addictive for fixed two variables.

Proof: Let consume f satisfies equation (1.4), define $F: X^3 \to Y$, for $\forall x, y, z \in X$,we have

$$F(x,y,z) = \frac{1}{12}[f(x,y+z) + f(x,y-z) - 2f(x,y) - 2f(x,z)].$$

According to Lemma 1.1 ,2.1, for fixed x, F is symmetric, quadratic for y, z, and have $f(x,y) = F(x,y,y)$. Let $B: X^2 \to Y$, and $B(x,y) = F(0,x,y)$, it is obviously that for $\forall y \in X$, we have $f(0,y) = B(y,y)$, and B is symmetric, quadratic. Define $A: X^3 \to Y$,for $\forall x, y, z \in X$, let $A(x,y,z) = F(x,y,z) - B(y,z)$, then it is clear that A is symmetric, quadratic for y, z.

For fixed $y, z \in X$, for all $x \in X$, we have that $A(x,y,z) = \frac{1}{12}[f(x,y+z) - f(0,y+z)] + \frac{1}{12}[f(x,y-z) - f(0,y-z)] - \frac{1}{6}[f(x,y) - f(0,y)] - \frac{1}{6}[f(x,z) - f(0,z)]$.

Because f is Jensen for the first variable, then according to article [1] A is addictive for the first variable. It is obvious that $f(x,y) = A(x,y,y) + B(y,y)$ .If there exist a function $B: X^2 \to Y$ and $A: X^3 \to Y$, such that $f(x,y) = A(x,y,y) + B(y,y)$ for $\forall x, y \in X$, and B is symmetric for fixed on variable, A is addictive for the first variable, and symmetric, quadratic for the other two variables.

Let $F(x,y,y) = A(x,y,y) + B(y,y)$, it is clear that F is symmetric for the last two variables, and $f(x,y) = F(x,y,y)$, by lemma1.1 we know that f is quartic for the second variable. Then

$$2f(\frac{x_1+x_2}{2}, y) = 2[A(\frac{x_1+x_2}{2}, y, y) + B(y,y)] = A(\frac{x_1+x_2}{2}, y, y) + A(\frac{x_1+x_2}{2}, y, y) + 2B(y,y)$$

$$= [A(x_1,y,y) + B(y,y)] + [A(x_2,y,y) + B(y,y)] = f(x_1,y) + f(x_2,y)$$

Similarly, we can deduced the following corollary.

Corollary 2.3 Let $f: X^2 \to Y$ be a function satisfies equation (1.4), $g: X \to Y$ is a quartic function, then for $h: X^2 \to Y$ satisfies $h(x,y) = f(x,y) + g(y)$, we claim that h is a Jensen-quartic function.

Proof: According to lemma 2.3, there exist a function $A: X^3 \to Y$ such that $f(x,y) = A(x,y,y)$

$\forall x, y \in X$, A is addictive to the first variable. Symmetric, quadratic for the other two variables.

By lemma1.1, we can deduces that there exists an function $B: X^2 \to Y$, and B is symmetric, biquadratic for all $x \in X$, $g(x) = B(x,x)$.

Then by Th. 2.3 we know $h(x,y) = f(x,y) + g(y) = A(x,y,y) + B(y,y)$ is a Jensen-quartic function.

3. The Ulam stability of (1.4) on Banach spaces

Though out this section, we assume that X is a real vector space, Y is a real Banach space, and we mainly using the so called "director method" to prove the stability of (1.4) on Banach spaces.

For a given mapping $f: X^2 \to Y$, we define the difference operator $Df: X^4 \to Y$

$$Df(x_1,x_2,y_1,y_2) = f(\frac{x_1+x_2}{2}, 2y_1+y_2) + f(\frac{x_1+x_2}{2}, 2y_1-y_2) - 2f(x_1,y_1+y_2) - 2f(x_1,y_1-y_2) -$$

$$12f(x_1,y_1) + 3f(x_1,y_2) - 2f(x_2,y_1+y_2) - 2f(x_2,y_1-y_2) - 12f(x_2,y_1) + 3f(x_2,y_2), \forall x_1,x_2,y_1,y_2 \in X$$

Theorem 3.1 Let $\varphi: X^3 \to [0,\infty)$ be a function such that for $\forall x_1,x_2,y_1,y_2 \in X$ satifying

$$\Phi(y_1,y_2) = \sum_{n=0}^{\infty} \frac{1}{81^{n+1}} [\varphi(0,0,3^n y_1, 3^n y_2) + 2\varphi(0,0,3^n y_1, 0)] < \infty. \qquad (3.1)$$

Suppose that $f: X^2 \to Y$ satisfying $\|Df(x_1,x_2,y_1,y_2)\| \le \varphi(x_1,x_2,y_1,y_2)$, (3.2)

and $\forall x \in X$, $f(x,0) = 0$. Then there exist a unique quartic mapping $Q: X \to Y$ such that

$$\|Q(y) - f(0,y)\| \le \Phi(y,y). \forall x, y \in X, \qquad (3.3)$$

Proof: In (3.2), we set $x_1 = x_2 = 0, y_1 = y_2 = y$, and obtain

$$\|Df(x_1,x_2,y_1,y_2)\| = \|f(0,3y) - 17f(0,y) - 4f(0,2y)\| \le \varphi(0,0,y,y);$$

In (3.2), put $x_1 = x_2 = 0, y_1 = y, y_2 = 0$, so $\|Df(x_1,x_2,y_1,y_2)\| = \|2f(0,2y) - 32f(0,y)\| \le \varphi(0,0,y,0)$;

Then it follows from the above inequality, we have

$$\|f(0,3y) - 81f(0,y)\| \le \varphi(0,0,y,y) + 2\varphi(0,0,y,0).$$

Let both sides of the inequality divide with 81, then we have

$$\left\|\frac{1}{81}f(0,3y)-f(0,y)\right\|\le\frac{1}{81}[\varphi(0,0,y,y)+2\varphi(0,0,y,0)].$$

Replace y by $3^n y$, and both sides divide with 81^n,we obtain

$$\left\|\frac{1}{81^{n+1}}f(0,3^{n+1}y)-\frac{1}{81^n}f(0,3^n y)\right\|\le\frac{1}{81^{n+1}}[\varphi(0,0,3^n y,3^n y)+2\varphi(0,0,3^n y,0)].$$

For any positive integer $m<n$, it is clearly that

$$\left\|\frac{1}{81^{n+1}}f(0,3^{n+1}y)-\frac{1}{81^m}f(0,3^m y)\right\|\le\sum_{i=m}^{n}\frac{1}{81^{i+1}}[\varphi(0,0,3^i y,3^i y)+2\varphi(0,0,3^i y,0)], \qquad (3.4)$$

Then by (3.1) we obtain $\{\frac{1}{81^n}f(0,3^n y)\}$ is a Cauchy series, by the completeness of Y, the Cauchy series is convergence. Define $Q:X\to Y$, $Q(x)=\lim_{n\to\infty}\frac{1}{81^n}f(0,3^n x),\forall x\in X$. In (3.2), put $x_1=x_2=0$, and replace y_1,y_2 by $3^n y_1,3^n y_2$, both sides divide with $\frac{1}{81^n}$, then we have:

$$\left\|\frac{1}{81^n}f(0,3^n(2y_1+y_2))+\frac{1}{81^n}f(0,3^n(2y_1-y_2))-\frac{4}{81^n}f(0,3^n(y_1+y_2))-\frac{4}{81^n}f(0,3^n(y_1-y_2))\right.$$

$$\left.-\frac{24}{81^n}f(0,3^n y_1)+\frac{6}{81^n}f(0,3^n y_2)\right\|\le\frac{1}{81^n}\varphi(0,0,3^n y_1,3^n y_2).$$

Let $n\to\infty$, then by (3.1) we know that , for $\forall y_1,y_2\in X$

$$Q(2y_1+y_2)+Q(2y_1-y_2)=4Q(y_1+y_2)+4Q(y_1-y_2)+24Q(y_1)-6Q(y_2),$$

Thus Q is a quartic function. Let $m=0,n\to\infty$ in (3.4) we have $\|Q(y)-f(0,y)\|\le\Phi(y,y)$.

Assume $Q':X\to Y$ is another quartic function satisfy the condition for $\forall y\in X$, then we have

$\|Q'(y)-Q(y)\|=\frac{1}{81^n}\|Q'(3^n y)-Q(3^n y)\|\le\frac{1}{81^n}\|Q'(3^n y)-f(0,3^n y)\|+\frac{1}{81^n}\|f(3^n y)-Q(3^n y)\|$

$\le\frac{2}{81^n}\Phi(3^n y,3^n y)=2\sum_{i=n}^{\infty}\frac{1}{81^{i+1}}[\varphi(0,0,3^i y,3^i y)+\varphi(0,0,3^i y,0)]\to 0$ .Then we proved the uniqueness.

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