Solutions for fractional boundary value problem with sign-changing Green's function

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ABSTRACT: In this paper, we study the solutions for the fractional boundary value problem $\begin{cases} {}^cD_t^\alpha\ u(t)+g(t)f(u(t))=0,\ t\in(0,1), \\ u(0)=0, u(1)=0, \end{cases}$ (1), where $1<\alpha<2$ is a real number. Define a new cone to solve the difficulty. And investigate solutions for the problem (1) with a sign-changing Green's function. And will research the sign-changing solutions.

1 INTRODUCTION

In these years, more and more authors study the fractional boundary value problems. In [1], the author consider the problem

$$\begin{cases} {}^cD^\alpha u(t)+f(u(t))=0, 0<t<1, \\ u(0)=u''(0)=0, u(1)=\lambda\int_0^1 u(s)ds, \end{cases}$$

and $2<\alpha<3, 0<\lambda<2, {}^cD^\alpha$ is the Caputo fractional derivate.

These papers are achieved when the corresponding Green's function are nonnegative, the question is that if the Green's function changes sign, how we study the problem. In [8], the author consider the problem with a sign-changing Green's function, the author define a new cone to solve the difficulty.

This paper will investigate solutions for the problem (1) with a sign-changing Green's function. And will research the sign-changing solutions.

2 PRELIMINARIES

For the BVP

$$\begin{cases} {}^cD_t^\alpha u(t)+\ g(t)f(u(t))=0,\ t\in(0,1), \\ u(0)=0, u(1)=0, \end{cases}$$

we have the following details.

From the definition of Caputo fractional derivate we know u is a solution of

$$\begin{cases} {}^cD_t^\alpha u(t)+y(t)=0,\ t\in(0,1), \\ u(0)=0, u(1)=0, \end{cases} \qquad (2)$$

when $u(t)=\int_0^1 G(t,s)y(s)ds,$ where

$$G(t,s)=\begin{cases} \dfrac{t(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, 0\le s\le t\le 1, \\ \dfrac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \qquad 0\le t\le s\le 1, \end{cases} \qquad (3)$$

If we choose $\alpha=\frac{2}{3}, s=\frac{1}{4}, t=\frac{1}{2}$ we can know that the (3) changes sign.

We define

$$G^{+}g(s)=\begin{cases}G(t,s)g(s), & G(t,s)g(s)\ \geq 0,\\ 0, & G(t,s)g(s)\ <0.\end{cases}$$

$$G^{-}g(s)=\begin{cases}-G(t,s)g(s), & G(t,s)g(s)\ \leq 0,\\ 0, & G(t,s)g(s)\ >0.\end{cases}$$

Let $E=C[0,1],\|u\|=\max\limits_{0\leq t\leq 1}|u|$, In order to use the cone theories, we need to define a new cone in E.

$\dot{K}=\{y(t)\in E:\ y(t)$ is a concave function$\}$,

$P=\{u\in \dot{K}:\ u(t)\geq 0,\ \int_0^1 u(t)\geq \dfrac{\beta}{M_1}\|u\|\}$,

where $\beta=\min\int_0^1 G(t,s)g(s)dt>0, M_1=\max\limits_{0\leq t,s\leq 1}|G(t,s)g(s)|$.

Let

$$\gamma=\begin{cases}+\infty, & Gg(s)\ \geq 0,\\ \dot{C}, & Gg(s)\ <0.\end{cases}$$

$$\dot{C}=\min_{t\in[0,1]}\frac{\int_0^1\ G^{+}g(s)ds}{\int_0^1\ G^{-}g(s)ds}$$

The operators K,F,A are as follows

$$\left(Ku\right)(t)=\int_0^t\frac{t(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds+$$

$$\int_t^1\frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds,$$

$(Fu)(t)=\ g(t)f(u(t))$,

$A=KF$.

Let

$$\beta_0=\lim_{u\to 0}\frac{g(t)f(u(t))}{u},\beta_\infty=\lim_{u\to\infty}\frac{g(t)f(u(t))}{u}\quad .$$

The following conditions are satisfied

$(H0)$ $f(x)\in C(R,R), f(\theta)=\theta, xf(x)>0$ for all $u\in R\setminus\{\theta\},\ g\in C([0,1],R)$;

$(H1)$ $\inf\limits_{u\in[0,+\infty]}f(u)=m\geq 0,\ \sup\limits_{u\in[0,+\infty]}f(u)=M\leq+\infty$, M/m $\leq\ \gamma$, when $m=0$, M/m $=+\infty$;

$(H2)$ $\lambda_{2n_0}<\beta_0<\lambda_{2n_0+1},\lambda_{2n_1}<\beta_\infty<\lambda_{2n_1+1};n_0,n_1$ are positive integers.

$(H3)$ $|g(s)f(u(s))|<\Gamma(\alpha)C_0$ for all u with $|u|\leq\ C_0$.

Lemma 2.1 Suppose that $(H0)$ *and* $(H1)$ *hold,*

then the operator $A:P\mapsto E$ *is completely continuous and* $A:P\mapsto P$.

Proof In case of $G\geq 0$, we have $(Au)(t)\geq 0$,
in case of $G<0$, we have

$$\left(Au\right)(t)=\int_0^1 G(t,s)g(s)f(u(s))ds$$
$$=\int_0^1 (G^+-G^-)g(s)f(u(s))ds$$
$$\geq \int_0^1 (mG^+g(s)-MG^-)g(s)ds$$
$$\geq m\int_0^1 G^+g(s)-\gamma G^-)g(s)ds\geq 0$$

On the other hand,

$$\int_0^1 (Au)(t)dt=\int_0^1 f(u(s))\int_0^1 G(t,s)g(s)dtds$$
$$\geq \beta\int_0^1 f(u(s))ds.$$

And

$$(Au)(t)\leq M_1\int_0^1 f(u(s))ds$$
$$\int_0^1 (Au)(t)dt\geq \frac{\beta}{M_1}(Au)(t)$$
$$\int_0^1 (Au)(t)dt\geq \frac{\beta}{M_1}\|(Au)(t)\| \quad \text{i.e.,}\quad A(P)\subseteq P.$$

Let $\Omega\in P$, that is $\exists M>0$, such that $\|u\|_\infty\leq M$

Define now $L=\max\limits_{0\leq t\leq 1, 0\leq u\leq M}|g(t)f(u(t))|+1$,Then

$$|(Au)(t)|\leq L\frac{1}{\Gamma(\alpha+1)},$$
$$|(Au)'(t)|\leq \frac{L}{\Gamma(\alpha)}+\frac{L}{(\alpha-1)\Gamma(\alpha-1)}:=N$$

Then we have

$$|(Au)(t_1)-(Au)(t_2)|\leq \int_{t_1}^{t_2}|(Au)'(s)|ds$$
$$\leq N(t_2-t_1)$$

Then the lemma is proofed.

Lemma 2.2 Suppose that $\left(H0\right)$ *holds, then the operator* A *is Frechet differentiable at* θ and ∞, *and* $A'(\theta)=\beta_0 K, A'(\infty)=\beta_\infty K$.

Proof It is from $\beta_0=\lim\limits_{u\to 0}\frac{g(t)f(u(t))}{u}$ that $\forall\varepsilon>0,\ \exists\delta>0,\ \forall 0<|u|<\delta,$ we have

$$\left|\frac{g(t)f(u(t))}{u}-\beta_0\right|<\varepsilon, \quad \text{that is}$$

$|g(t)f(u(t))-\beta_0 u|<\varepsilon|u|$, then by $\left(H0\right)$, we have

$$|(Au-A\theta-\beta_0 Ku)(t)|\leq \frac{1}{\alpha\Gamma(\alpha)}\varepsilon\|u\|$$

That implies

$$\lim_{\|u\|\to 0}\frac{\|Au-A\theta-\beta_0 Ku\|}{\|u\|}=0.$$

This means $A'(\theta)=\beta_0 K$.

We can prove $A^{'}(\infty)=\beta_\infty K$ as the same way.

Lemma 2.3 Let β be a positive number, then the sequence of the positive eigenvalues of the operator βK is $\frac{\beta}{\lambda_1}>\frac{\beta}{\lambda_2}>...>\frac{\beta}{\lambda_{n_\alpha}}$. And if we suppose that $E^{(1)}_{\alpha,2}(-\lambda_k)\neq 0$, then the eigenvalue $\frac{\beta}{\lambda_k}$ will have algebraic multiplicity one, $1\le \lambda_k \le n_\alpha$.

Proof From reference [6], we know that $\frac{1}{\lambda}$ is the positive eigenvalue of the operator K if and only if $E_{\alpha,2}(-\lambda)=0$. Therefore, the eigenvalue of the operator βK is $\frac{\beta}{\lambda_1}>\frac{\beta}{\lambda_2}>...>\frac{\beta}{\lambda_{n_\alpha}}$ and the eigenfunction corresponding to the eigenvalue $\frac{\beta}{\lambda_n}$ is

$u_n(t)=CtE_{\alpha,2}(-\lambda_n t^\alpha)$.

$$\dim\ker(\frac{\beta}{\lambda_n}I-\beta K)=\dim\ker(I-\lambda_n K)=1,$$

we need to show $\ker(I-\lambda_n K)=\ker(I-\lambda_n K)^2$, Let $u\in\ker(I-\lambda_n K)^2$, then if $u\notin\ker(I-\lambda_n K)$, there exist a nonzero constant C such that $(I-\lambda_n K)(u)=CtE_{\alpha,2}(-\lambda_n t^\alpha)$. By direct computation, we have

$$\begin{cases} {}^cD_t^\alpha\ u(t)+\lambda_n\ u(t)=-Ct\lambda_n E_{\alpha,2}(-\lambda_n t^\alpha), \\ u(0)=0, u(1)=0, \end{cases}$$

From the Laplace transform,

$$L\{u(t)\}=u'(0)\frac{s^{\alpha-2}}{s^\alpha+\lambda_n}-\frac{C\lambda_n s^{\alpha-2}}{(s^\alpha+\lambda_n)^2}.$$

$$u(t)=u'(0)tE_{\alpha,2}(-\lambda_n t^\alpha)-C\lambda_n t^{\alpha+1}E^{(1)}_{\alpha,2}(-\lambda_n t^\alpha)$$

Let $u(1)=0$, then we get $E^{(1)}_{\alpha,2}(-\lambda_n)=0$. Which is a contradiction.

Then the lemma is proofed.

Lemma 2.4 Suppose that $(H2)$ holds, $u\in P\setminus\{\theta\}$ is a solution of (1), then $u\in \overset{\circ}{P}$.

Proof: $u(t)$ is a concave function, $u(0)=0, u(1)=0$,

so we have

$u'(0)>0,\ u(t)\ge 0.$

From $u'(0)>0$ we know $\exists\varepsilon>0,\tau_1>0$,

$$u'(t)>\tau_1,\quad \forall t\in[0,\varepsilon].$$

From $u(t)\ge 0$, we know $\exists\tau_2>0$

$$u(t)>\tau_2,\quad \forall t\in[\varepsilon,1].$$

令 $\tau=\min(\tau_1,\tau_2)$, then

$x(t)\ge 0,\ t\in[0,1].$

So $B(u,\tau)\subset P, u\in\overset{\circ}{P}.$

Lemma 2.5 Suppose that $(H0)(H2)$ hold, then

(1) *there exist* $C_0 > r_0 > 0$ *such that* $\forall 0 < r \le r_0$,
$i(A, P \cap B(\theta, r), P) = 0, i(A, -P \cap B(\theta, r), -P) = 0;$ (4) (2)*there exist* $R_0 > C_0$ *such that* $\forall R \ge R_0$,
$i(A, P \cap B(\theta, R), P) = 0, i(A, -P \cap B(\theta, R), -P) = 0;$ (5) Proof We only proof (4), By $(H0)$ we know

By lemma 2.1 and lemma 2.2 we know

$$\frac{\beta_0}{\lambda_1}(\frac{\beta_0}{\lambda_1} > 1)$$

is the eigenvalue of $\beta_0 K$, and the corresponding eigenfunction is
$u(t) = CtE_{\alpha,2}(-\lambda_1 t^{\alpha}),$
The smallest zero of $E_{\alpha,2}(-x) = 0$ is λ_1 ,
$E_{\alpha,2}(-\lambda_1 t^{\alpha}) \ne 0, \forall t \in (0,1).$
We can choose the suitable C to ensure $u(t) \ge 0.$
By the lemma in [2] we know (4) is proofed.

3 PROOF OF MAIN RESULT

Theorem 3.1 Suppose that $(H0)-(H3)$ *hold. And* $E^{(1)}_{\alpha,2}(-\lambda_n) \ne 0,$ *where* $n = 1, 2, \ldots \max(2n_0, 2n_1)$, *then* (1) *has at least two sign-changing solutions.*

Proof By $(H3)$, we have

$$|(Au)(t)| = \quad < \Gamma(\alpha) C_0 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds < C_0$$

Therefore $\|Au\| \le C_0, \forall \|u\| = C_0.$
Then we have

$$i\ (A, P \cap B(\theta, C_0), P) = 1, \quad (6)$$

$$\deg\ (I - A, B(\theta, C_0), \theta) = 1. \quad (7)$$

By lemma2.3, lemma2.5, and $(H2)$, we have $\exists 0 < r_1 < r_0$ such that

$$\deg(I - A, B(\theta, r_1), \theta) = (-1)^{2n_0} = 1. \quad (8)$$

Similarly, by lemma2.3, lemma2.8 and $(H2)$, $\exists R_1 \ge R_0$ such that

$$\deg(I - A, B(\theta, R_1), \theta) = (-1)^{2n_1} = 1. \quad (9)$$

By lemma 2.5, we have

$$i(A, P \cap B(\theta, r_1), P) = 0, \quad (10)$$

$$i(A, P \cap B(\theta, R_1), P) = 0, \quad (11)$$

By (6)(10)(11), we have

$$i(A, P \cap (B(\theta, R_1) \setminus \overline{B(\theta, C_0)}), P) = 0 - 1 = -1, \quad (12)$$

$$i(A, P \cap (B(\theta, C_0) \setminus \overline{B(\theta, r_1)}), P) = 1 - 0 = 1, \quad (13)$$

A has at least two fixed point
$u_1 \in P \cap (B(\theta, R_1) \setminus \overline{B(\theta, C_0)}),$
$u_2 \in P \cap (B(\theta, C_0) \setminus \overline{B(\theta, r_1)}).$
u_1 and u_2 are two positive solutions of (1), and
$r_1 < \|u_1\| \le C_0 < \|u_2\| \le R_1.$
Similarly, we get (1) has two negative solutions
$-u_3 \in P \cap\ (B(\theta, R_1) \setminus \overline{B(\theta, C_0)}),$

$-u_4 \in P \cap \ (B(\theta, C_0) \setminus \overline{B(\theta, r_1)})$.

And $\quad r_1 < \|u_3\| \le C_0 < \|u_4\| \le R_1$

By lemma2.4, and lemma in [13], there exist four open subsets O_1, O_2, O_3, O_4 of E such that

$O_1 \subset P \cap (B(\theta, R_1) \setminus \overline{B(\theta, C_0)})$,

$O_2 \subset P \cap (B(\theta, C_0) \setminus \overline{B(\theta, r_1)})$,

$O_3 \subset -P \cap (B(\theta, C_0) \setminus \overline{B(\theta, r_1)})$,

$O_4 \subset -P \cap (B(\theta, R_1) \setminus \overline{B(\theta, C_0)})$

Then

$$\deg(I - A, O_1, \theta) = -1, \deg(I - A, O_4, \theta) = -1, \qquad (14)$$

$$\deg(I - A, O_2, \theta) = 1, \deg(I - A, O_3, \theta) = 1, \qquad (15)$$

By (7)(8)(15), we have

$$\deg(I - A, B(\theta, C_0) \setminus (\overline{O_2} \cup \overline{O_3} \cup \overline{B(\theta, r_1)}), \theta) = -2 \quad (16)$$

By (7)(9)(14), we have

$$\deg(I - A, B(\theta, R_1) \setminus (\overline{O_1} \cup \overline{O_4} \cup \overline{B(\theta, C_0)}), \theta) = 2 \quad (17)$$

By (16) (17) we know the problem (1) has two sign-changing solutions. The theorem is proofed.

3 CONCLUSION

We research the solutions for the fractional boundary value with sign-changing Green’s function in this paper. Define a new cone to solve the difficulty. And investigate solutions for the problem (1) with a sign-changing Green’s function. And will research the sign-changing solutions.

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