The Positive Lyapunov Exponent Of The Matrix With The Exponential Rotation

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Abstract. This paper studies the Lyapunov exponent defined by the matrix with the exponential rotation. The author applies the theory of subharmonic functions to prove that if the coupling number is big enough, then the Lyapunov exponent is positive.

1. Introduction

In this paper, we consider the Lyapunov exponent defined by the following matrix:

$$(1)\qquad A(x,E) := \begin{pmatrix} \lambda v_{11}(x)-E & v_{12}(x) & \cdots & v_{1m}(x) \\ v_{21}(x) & v_{22}(x) & \cdots & v_{2m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1}(x) & v_{m2}(x) & \cdots & v_{mm}(x) \end{pmatrix},$$

where every $v_{ij}(x), i, j = 1, 2\cdots, m,$ is an analytic function on $\mathbb{T} := \mathbb{R}\backslash\mathbb{Z}.$ Then

$$(2)\qquad M_n(x,E) = \prod_{j=n-1}^{0} A(T^j(x),E) = \prod_{j=n-1}^{0} \begin{pmatrix} \lambda v_{11}(T^j(x))-E & v_{12}(T^j(x)) & \cdots & v_{1m}(T^j(x)) \\ v_{21}(T^j(x)) & v_{22}(T^j(x)) & \cdots & v_{2m}(T^j(x)) \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1}(T^j(x)) & v_{m2}(T^j(x)) & \cdots & v_{mm}(T^j(x)) \end{pmatrix}$$

is called the transfer matrix of (1) and here T is the exponential rotation:

$$(3)\qquad T^j(x) = x + \omega^j,$$

where ω is a irrational number. Now we can define the Lyapunov exponent:

$$(4)\qquad L(E) = \liminf_{n\to\infty} \frac{1}{n}\int_{\mathbb{T}} \log\|M_n(x,E)\|dx,$$

where $\|M_n(x,E)\| = \max_{\|v\|_2=1}\|M_n(x,E)v\|_2$. In this paper, we will show that this Lyapunov exponent is positive if the coupling number is big enough. For the detail, please see the main theorem at the end of this paper.

Actually, this question comes from some famous discrete operators, such like the Schrodinger operator, the Jacobi operator, the extended Harper's model and so on, which have many applications in Physics. Also, the property of matrix multiplication is a significant topic in computer theory. So, similar questions were studied in [1], [2] and [5].

2. Positive Lyapunov Exponent

In this section, we show the proof of the main theorem. First, we recommend the following lemmas from [2] and [4], which will be applied in the proof of the positive Lyapunov exponent:

Lemma 2: For all $0<\delta<\rho,$ there is an ϵ such that

$$(5)\qquad \inf_{E_1}\sup_{\frac{\delta}{2}<y<\delta}\inf_{x\in[0,1]}|v(x+iy)-E_1| > \epsilon.$$

Lemma 3: Let $u:\Omega\to[-\infty,+\infty)$ be an upper semi-continuous function. Then u(z) is a subharmonic function on $\Omega,$ if and only if for any Jordan subdomain Ω' satisfying $\overline{\Omega'}\subset\Omega$ and any $z\in\Omega',$ it has

$$u(z)\le\int_{\partial\Omega'}u(\zeta)d\mu_\zeta(z,\partial\Omega',\Omega'),$$

where $\mu(z, \partial\Omega', \Omega')$ is the harmonic measure of $\partial\Omega'$ at $z \in \Omega'$.

Now without loss of generality, let $\lambda > 0$. We all know that if v is real analytic function on $\mathbb{T}$, then there exists some $\rho_v > 0$ such that

$$v(x) = \sum_{k\in\mathbb{Z}} \hat{v}(k)e^{2\pi ikx}, \quad \text{with} \quad |\hat{v}(k)| \lesssim e^{-\rho_v|k|}. \tag{6}$$

So, it has a holomorphic extension

$$v(z) = \sum_{k\in\mathbb{Z}} \hat{v}(k)e^{2\pi ikz}$$

to the strip $|\mathrm{Im}z| < \frac{\rho_v}{10}$, satisfying

$$|v(z)| \leq \sum_{k\in\mathbb{Z}} |\hat{v}(k)|e^{2\pi|k||\mathrm{Im}z|} < \sum_{k\in\mathbb{Z}} e^{-\rho_v|k|}e^{\rho_v|k|\frac{\pi}{10}} < C_v. \tag{7}$$

Thus, for any $1 \leq i, j \leq m$, it has

$$v_{ij}(x) = \sum_{k\in\mathbb{Z}} \hat{v}_{ij}(k)e^{2\pi ikx}, \quad \text{with} \quad |\hat{v}_{ij}(k)| \lesssim e^{-\rho_{ij}|k|},$$

Then define

$$C_{ij} = \sup_{|\mathrm{Im}z|\leq\frac{\rho_{ij}}{10}} |v_{ij}(z)|, \ 1 \leq i, \ j \leq m, \ \ C = \max_{i,j} C_{ij} \text{ and } \rho = \min_{i,j} \rho_{ij}.$$

Then, by uniformly hyperbolic, we can assume that $|E| < mC\lambda$ in the following paper. Thus, $M_n(z, E)$ defined in (2), is analytic on $|\mathrm{Im}z| < \frac{\rho}{10}$ with fixed ω and E, and $\|M_n(z, E)\| \leq (2mC\lambda)^n$. So, define

$$u_n(z) := \frac{1}{n}\log\|M_n(z, E)\|,$$

which is a subharmonic function on $|\mathrm{Im}z| < \frac{\rho}{10}$, upper bounded by $\log[2mC\lambda]$. Then the Lyapunov exponent becomes

$$L(E) = \liminf_n L_n(E),$$

where

$$L_n(E) = \int_{\mathbb{T}} u_n(x)dx.$$

Fix $0 < \delta \ll \rho$ and ϵ satisfying Lemma2. Define

$$\lambda_0 = 100mC\epsilon^{-100} > 0$$

and let $\lambda > \lambda_0 > 0$. Then, for fixed E, there is $\frac{\delta}{2} < y_0 < \delta$ such that

$$\inf_{x\in[0,1]} \left|v_{11}(x + iy_0) - \frac{E}{\lambda}\right| > \epsilon.$$

Thus, by periodic,

$$\inf_{x\in\mathbb{R}} |\lambda v_{11}(x + iy_0) - E| > \lambda\epsilon > 100mC\epsilon^{-99} > 100mC. \tag{8}$$

Define

$$M_{n-1}(iy_0, E)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} w_1^{n-1} \\ w_2^{n-1} \\ \vdots \\ w_m^{n-1} \end{pmatrix},$$

Then

$$\begin{pmatrix} w_1^n \\ w_2^n \\ \vdots \\ w_m^n \end{pmatrix} = \begin{pmatrix} \lambda v_{11}(iy_0+\omega^n)-E & v_{12}(iy_0+\omega^n) & \cdots & v_{1m}(iy_0+\omega^n) \\ v_{21}(iy_0+\omega^n) & v_{22}(iy_0+\omega^n) & \cdots & v_{2m}(iy_0+\omega^n) \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1}(iy_0+\omega^n) & v_{m2}(iy_0+\omega^n) & \cdots & v_{mm}(iy_0+\omega^n) \end{pmatrix} \begin{pmatrix} w_1^{n-1} \\ w_2^{n-1} \\ \vdots \\ w_m^{n-1} \end{pmatrix}$$

$$(9) \qquad = \begin{pmatrix} (\lambda v_{11}[iy_0+\omega^n]-E)w_1^{n-1} + \sum_{j=2}^m v_{1j}[iy_0+\omega^n]w_j^{n-1} \\ \sum_{j=1}^m v_{2j}[iy_0+\omega^n]w_j^{n-1} \\ \vdots \\ \sum_{j=1}^m v_{mj}[iy_0+\omega^n]w_j^{n-1} \end{pmatrix}.$$

Now we use induction to show that

$$(10) \qquad |w_1^n| \geq |w_j^n|,\ j=2,\cdots,m, \text{ and } |w_1^n| \geq (\lambda\epsilon - mC)|w_1^{n-1}| \geq (\lambda\epsilon - mC)^n,\ \ n \geq 1.$$

Because $w_1^0 = 1,\ w_j^0 = 0,\ j=2,\cdots,m,$ so it yields

$$|w_1^1| = \lambda\epsilon > 100mC,\ \ |w_j^1| < C,\ j=2,\cdots,m,$$

which satisfy (10) for n=1. Then let n=t with

$$(11) \qquad |w_1^t| \geq |w_j^t|,\ j=1,\cdots,m, \text{ and } |w_1^t| > (\lambda\epsilon - mC)|w_1^{t-1}| > (\lambda\epsilon - mC)^t.$$

Due to (9) and (11), it has

$$\begin{aligned} |w_1^{t+1}| &\geq (\lambda\epsilon - mC)\,w_1^t > (\lambda\epsilon - mC)^{t+1}, \\ |w_j^{t+1}| &\leq mC|w_1^t| < 99mC|w_1^t| \leq (\lambda\epsilon - mC)|w_1^t| \leq |w_1^{t+1}|,\ j=2,\cdots,m, \end{aligned}$$

which also satisfy (10) for n=t+1. Thus, the expression (10) holds for any $n \geq 1.$ Then

$$\|M_n(iy_0,E)\| > (\lambda\epsilon - mC)^n \text{ and } u_n(iy_0) > \log(\lambda\epsilon - mC).$$

Write $\mathbb{H}_s$ for the strip $\{z = x+iy : 0 < y < \frac{\rho}{10}\}$ and denote $\mu_s(iy_0, E_s, \mathbb{H}_s)$ the harmonic measure of E_s at $iy_0 \in \mathbb{H}_s,$ where $E_s \subset \partial\mathbb{H}_s = \mathbb{R}\bigcup[y=\frac{\rho}{10}].$ Thus, by [3],

$$\mu_s[y=\frac{\rho}{10}] = \frac{10\pi y_0}{\pi\rho} < \frac{10\delta}{\rho} \text{ and } \frac{d\mu_s(x)}{dx}\Big|_{y=0} < \frac{y_0}{x^2+y_0^2}.$$

So, by subharmonic and Lemma 3, it yields

$$\begin{aligned} \log(\lambda\epsilon - mC) < u_n(iy_0) &\leq \int_{[y=0]\bigcup[y=\frac{\rho}{10}]} u_n(z)\mu_s(dz) \\ &= \int_{y=0} u_n(x)\mu_s(dx) + \int_{y=\frac{\rho}{10}} u_n(x+iy)\mu_s(dx) \\ &\leq \int_{\mathbb{R}} u_n(x)\frac{y_0}{x^2+y_0^2}dx + \frac{10\delta}{\rho}\left[\sup_{y=\frac{\rho}{10}} u_n(x+iy)\right] \\ &\leq \int_{\mathbb{R}} u_n(x)\frac{y_0}{x^2+y_0^2}dx + \frac{\bar{C}\delta}{\rho}\log\lambda. \end{aligned}$$

Then

$$\begin{aligned} L_n(E) = \int_0^1 u_n(\theta)d\theta &\geq \frac{y_0}{2}\int_0^1 u_n(\theta)\left(\sum_{k\in\mathbb{Z}}\frac{y_0}{y_0^2+(\theta+k)^2}\right)d\theta \\ &\geq \frac{y_0}{2}\left(\log(\lambda\epsilon - mC) - \frac{10\delta}{\rho}\log(2mC\lambda)\right) \\ (12) \qquad &\geq \frac{\delta}{4}\left(\frac{1}{2}\log\lambda - \frac{1}{100}\log\lambda_0\right) \\ &\geq c\log\lambda,\ \forall n \geq 1. \end{aligned}$$

where the inequality (12) comes from the setting of λ_0 and $\delta \ll \rho.$ Note that the small constant c depends on all $v_{ij}.$ Thus, it has

$$L(E) = \liminf_n L_n(E) \geq c\log\lambda,$$

which proves the following main theorem:
Main Theorem: There exists $\lambda_0 > 0$ such that for any irrational ω, if the coupling number $|\lambda| > \lambda_0$, then

$$L(E) \geq c \log|\lambda| \quad for\ all\ \ E,$$

where λ_0 and c depend only on $v_{ij}, i, j = 1, 2, \cdots, m.$

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