Stiffness Matrix of Timoshenko Beam Element with Arbitrary Variable Section

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Abstract: Timoshenko beam with variable section is widely used for the sake of good mechanical behavior and economic benefit. In order to improve analytical accuracy, stiffness matrix of Timoshenko beam element with arbitrary section was founded. According to the relationship between geometrical deformation and element internal force, by integral of sectional curvature, shearing strain and axial strain, stiffness matrix of the Timoshenko beam element was derived. Then, analytical program was developed, which was proved exact in comparison with theoretical solution.

Introduction

Beams with variable section was widely used in large-span bridges which not only reduce the self-weight of structure significantly to improve spanning capacity, but also make full advantage of material. So accurate analysis of variable section beam makes great sense. In current studies, researchers usually adopt one of three methods to analyze beam element with variable section[1]. First method, beam element with variable section is considered as uniform beam based on average equivalent method[2-3]. Taking average of sections between two ends of the beam is one kind of average equivalent method. In the second method, stiffness matrix of beam element with variable section is founded by equilibrium equation[4]. In other studies, principle of minimum potential energy is adopted to establish the stiffness matrix of the beam element with variable section[5-8].

Based on the current studies, the relationship between geometrical deformation and element internal force was established by integral of sectional curvature, shearing strain and axial strain. So the stiffness matrix of the Timoshenko beam element was derived directly from the relationship mentioned above. The shape function of element with arbitrary variable section was easily derived from the study, which could be applied to further studies. Then, program was developed based on the equation derived above. At last, numerical calculation about a cantilever beam structure was carried out, which prove that stiffness matrix of beam element with arbitrary variable section is correct.

Calculation model

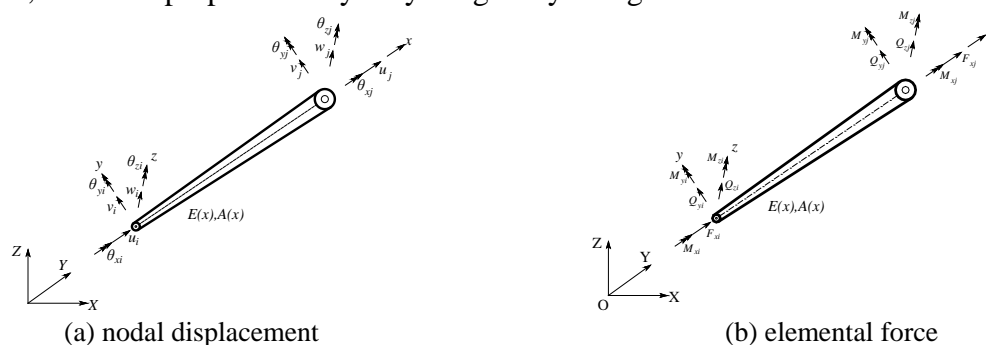In practical, sectional properties may vary irregularly along the element as shown in Fig.1.

(a) nodal displacement   (b) elemental force
Fig.1   Spatial Timoshenko beam element with arbitrary variable section

In Fig.1(a), u, v and w are displacements along x, y and z respectively; θ_x, θ_y, θ_z are rotation angles around x, y and z respectively; subscript i and j indicate node i and j. In Fig.1(b), M_x is torque around x; M_y and M_z are the moment around y and z respectively; F_x is axial force; F_y and F_z are shear force along y and z respectively.

Variable functions of arbitrary section properties are assumed in the forms expressed in Eq.1,

$$I(x) = I_i f_I(x) \qquad A(x) = A_i f_A(x) \qquad A_s(x) = A_{si} f_{A_s}(x) \tag{1}$$

where, $f_I(x)$、$f_A(x)$、$f_{A_s}(x)$ are distribution functions of inertial moment, axial area and shear area of beam element with arbitrary section respectively. Four variables are defined as shown in Eq.2~Eq.3.

$$L_{k0}(x) = \int_0^x \frac{1}{f_k(u)} du \qquad L_{k1}(x) = \int_0^x \frac{x}{f_k(u)} du \tag{2}$$

$$L_{k0r}(x) = \int_0^x L_{k0}(u) du \qquad L_{k1r}(x) = \int_0^x L_{k1}(u) du \tag{3}$$

According to mechanical analysis of beam element as shown in Fig.1, sectional internal force of any position along the element can derived from formula as shown in Eq.4.

$$\mathbf{F}_I(x) = \mathbf{H}_F \mathbf{F}_I^i + \mathbf{F}_{Ipy} \tag{4}$$

where, $\mathbf{F}_I^i$ and $\mathbf{F}_I(x)$ are respectively the section internal forces at section i and the section which is x away from the section i; $\mathbf{H}_F$ is transfer matrix of section internal force. $\mathbf{F}_I^i$, $\mathbf{F}_I(x)$ and $\mathbf{H}_F$ can expressed in Eq.5~Eq.7.

$$\mathbf{F}_I(x) = \begin{bmatrix} N(x) & Q_y(x) & Q_z(x) & T(x) & M_y(x) & M_z(x) \end{bmatrix}^T \tag{5}$$

$$\mathbf{F}_I^i = \begin{bmatrix} N_i & Q_{yi} & Q_{zi} & T_i & M_{yi} & M_{zi} \end{bmatrix}^T \tag{6}$$

$$\mathbf{H}_F = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -x & 0 & -1 & 0 \\ 0 & x & 0 & 0 & 0 & -1 \end{bmatrix}^T \tag{7}$$

The stiffness matrix of beam element in plane xoy can be first derived, the equivalent value in plane xoz can be similarly derived. Subscript y in variables defined above facilitate derivation. Then section internal force vector in plane xoy can be expressed as Eq.8

$$\mathbf{F}_{Iy}(x) = \mathbf{H}_{Fy}(x) \mathbf{F}_{Iy}^i + \mathbf{F}_{Ipy} \tag{8}$$

Derivation of stiffness matrix

Axial and torsional stiffness

Axial strain and twist ratio of beam element with variable section is not constant, the axial displacement can be derived by integral as shown in Eq.9,

$$N_i = \frac{EA_i(u_i - u_j)}{L_{A0}(l)} \tag{9}$$

where, N_i is the axial force in node i.

Axial force at section j could be derived from N_i based on the static equilibrium equation. Likewise, torque is derived from formula as shown in Eq.10.

$$T_i = \frac{GI_{xi}(\theta_{xi} - \theta_{xj})}{L_{I_x0}(l)} \tag{10}$$

Bending and shearing stiffness

Total vertical displacement includes two parts as shown in Eq.11,

$$v = v_{by} + v_{sy} \tag{11}$$

where, v_{by} is the displacement induced by bending moment, v_{sy} is the displacement induced by shearing force.

Firstly, taking vector $\boldsymbol{\delta}_y$ as section response as shown in Eq.12,

$$\boldsymbol{\delta}_y(x) = \begin{bmatrix} v_y(x) & \theta(x) \end{bmatrix}^{\mathrm{T}} \tag{12}$$

According to mechanics of materials, relationship between displacements and section internal force was established as shown in Eq.13,

$$\mathbf{F}_{ly}^i = \mathbf{H}_{1y}\mathbf{H}_{2y}\mathbf{d}_y^e \tag{13}$$

where, $\mathbf{F}_{ly}^i = \begin{bmatrix} Q_{yi} & M_{zi} \end{bmatrix}^{\mathrm{T}}, \mathbf{d}_y^e = \begin{bmatrix} v_{yi} & \theta_{zi} & v_{yj} & \theta_{zj} \end{bmatrix}^{\mathrm{T}}, \mathbf{H}_{1y} = \begin{bmatrix} \mathbf{N}_{dy}(l) \end{bmatrix}^{-1}$

$$\mathbf{H}_{2y} = \begin{bmatrix} -1 & -l & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{14}$$

$$\mathbf{N}_{dy}(x) = \frac{1}{EI_{zi}} \begin{bmatrix} L_{I_z 1r}(x) - \dfrac{l^2 b_y}{12} L_{A_{sy}0}(x) & -L_{I_z 0r}(x) \\ L_{I_z 1}(x) & -L_{I_z 0}(x) \end{bmatrix} \tag{15}$$

Stiffness matrix was further derived from Eq.9, Eq.10 and Eq.13.

For uniform beam element, variable functions described in Eq.1 equal 1. Parameters defined in Eq.2~Eq.3 will be simplified as shown in Eq.16~Eq.17.

$$L_{k0}(x) = x \qquad L_{k1}(x) = \frac{1}{2}x^2 \tag{16}$$

$$L_{k0r}(x) = \frac{1}{2}x^2 \qquad L_{k1r}(x) = \frac{1}{6}x^3 \tag{17}$$

Then, stiffness matrix of uniform beam element was derived.

Example analysis

A FEA program was developed based on the equation derived above. A cantilever beam as shown in Fig.2 is selected as the example object. The beam height varies linearly along the axes. Load and dimension of the structure are listed in Table1.

Table 1. Calculation parameter

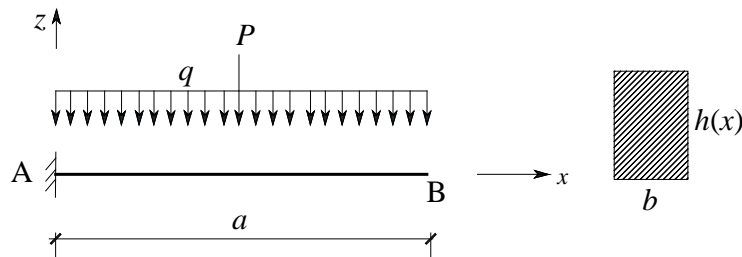| E/MPa	a/m	b/m	h_A/m	h_B/m	q/(kN/m)	P/(kN)
3.0e4	10	0.25	1.4	0.2	5	10

Fig.2. Cantilever beam with rectangular section

According to structural mechanics，vertical displacement of section B v_B=-9.7838mm，and sectional rotation angle θ_B=0.001798rad. Displacement and rotation angle in section B calculated from different models are showed in Fig.3. Lateral axis shows number of equivalent section applied to simulate the structure by average equivalent method.

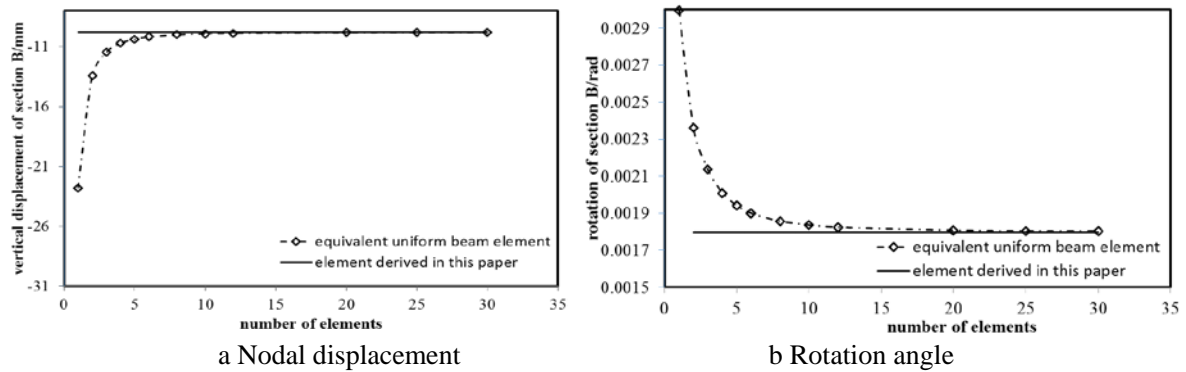a Nodal displacement   b Rotation angle

Fig.3. Response of section B calculated from different models

As shown in Fig.3, response calculated from model with only one element derived in this paper is exact compared to theoretical solution. In the contrary, more elements are needed to get exact solution for models based on average equivalent method.

Conclusion

1）Stiffness matrix of Timoshenko beam element with variable section was derived by geometrical analysis and element equilibrium condition.

2）Based on the theoretical study, the calculation program was developed. Numerical calculation was carried on a cantilever beam structure which shows that the Timoshenko beam element with variable section is correct.

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