Positive solutions of some nonlocal fourth-order boundary value problem with dependence on the first order derivative

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Abstract. In this paper, By the use of a new fixed point theorem and the nonlocal BVP Green function. The existence of at least one positive solutions for the nonlocal fourth-order boundary value problem with the first order derivative

$$\begin{cases} u^{(4)}(t) + Au''(t) = \lambda f(\mathrm{t}, u(t), u'(t)), 0 < t < 1 \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds \end{cases}$$

is considered, where f is a nonnegative continuous function and $\lambda > 0, 0 < A < \pi^2$, $p, q \in L[0,1]$, $p(s) \geq 0, q(s) \leq 0$.

1. Introduction

The deformation of an elastic beam in equilibrium state, whose two ends are simply supported, can be described by a fourth-order ordinary equation boundary value problem. Owing to its significance in physics, it has been studied by many authors using nonlinear alternatives of Leray-Schauder, the fixed point index theory, the Krasnosel'skii's fixed point theorem and the method of upper and lower solutions, see for example [1-4，6-10] .

Recently, there has been much attention focused on the question of positive solution of fourth-order differential equation with one or two parameters. For example, Li [6] investigated the existence of positive solutions for the fourth-order boundary value problem. Ma [9] studied the existence of symmetric positive solutions of the nonlocal fourth-order boundary value problem. Bai [3] studied the existence of positive solutions of the nonlocal fourth-order boundary value problem by the use of the Krasnosel'skii's fixed point theorem. All the above works were done under the assumption that the first order derivative u' is not involved explicitly in the nonlinear term f .

In this paper, we are concerned with the existence of positive solutions for the fourth-order three-point boundary value problem

$$\begin{cases} u^{(4)}(t) + Au''(t) = \lambda f(\mathrm{t}, u(t), u'(t)), 0 < t < 1 \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds \end{cases} \tag{1}$$

The following conditions are satisfied throughout this paper:

(H_1) $\lambda > 0, 0 < A < \pi^2$;

(H_2) $f:[0,1]\times[0,\infty)\times R \to [0,\infty)$ is continuous, $p, q \in L[0,1]$, $p(s) \geq 0, q(s) \leq 0$,

$\int_0^1 p(s)ds < 1, \int_0^1 q(s)\sin\sqrt{A}sds + \int_0^1 q(s)\sin\sqrt{A}(1-s)ds < \sin\sqrt{A}$.

2. The preliminary lemmas

Suppose $Y = C[0,1]$ be the Banach space equipped with the norm $\|u\|_0 = \max\limits_{t\in[0,1]}|u(t)|$.

Let λ_1, λ_2 be the roots of the polynomial $P(\lambda) = \lambda^2 + A\lambda$, namely, $\lambda_1 = 0, \lambda_2 = -A$. By ($H_1$) it is easy to see that $-\pi^2 < \lambda_2 < 0$.

Let $Q_i(t,s)(i=1,2)$ be the Green ' s function of the linear boundary value problem：

$$\begin{cases} -u''(t) + \lambda_1 u(t) = 0, 0 < t < 1 \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds \end{cases}$$

Then, carefully calculation yield:

$$Q_1(t,s) = G_1(t,s) + \frac{1}{1-\int_0^1 p(x)dx}\int_0^1 G_1(s,x)p(x)dx \,;\quad G_1(t,s) = \begin{cases} s(1-t), 0 \le s \le t \le 1 \\ t(1-s), 0 \le t \le s \le 1 \end{cases}$$

$$\begin{cases} -u''(t) + \lambda_2 u(t) = 0, 0 < t < 1 \\ u(0) = u(1) = \int_0^1 q(s)u(s)ds \end{cases}$$

Then, carefully calculation yield:

$$Q_2(t,s) = G_2(t,s) + \frac{\sin\sqrt{A}t + \sin\sqrt{A}(1-t)}{\sin\sqrt{A} - \int_0^1 \sin\sqrt{A}xq(x)dx - \int_0^1 \sin\sqrt{A}(1-x)q(x)dx}\int_0^1 G_2(s,x)q(x)dx$$

$$G_2(t,s) = \begin{cases} \dfrac{\sin\sqrt{A}s\sin\sqrt{A}(1-t)}{\sqrt{A}\sin\sqrt{A}}, 0 \le s \le t \le 1 \\ \dfrac{\sin\sqrt{A}t\sin\sqrt{A}(1-s)}{\sqrt{A}\sin\sqrt{A}}, 0 \le t \le s \le 1 \end{cases}$$

Denote1: $\omega_1 = \dfrac{1}{1-\int_0^1 p(x)dx}$, $\omega_2(t) = \dfrac{\sin\sqrt{A}t + \sin\sqrt{A}(1-t)}{\sin\sqrt{A} - \int_0^1 \sin\sqrt{A}xq(x)dx - \int_0^1 \sin\sqrt{A}(1-x)q(x)dx}$

Lemma 2.1: Suppose (H_1) (H_2) hold. Then for any $y(t) \in C[0,1]$, BVP

$$\begin{cases} u^{(4)}(t) + Au''(t) = y(t), 0 < t < 1 \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds \end{cases} \quad (2)$$

the unique solution

$$u(t) = \int_0^1\int_0^1 Q_1(t,s)Q_2(s,\tau)y(\tau)d\tau ds \quad (3)$$

where $Q_1(t,s) = G_1(t,s) + \omega_1\int_0^1 G_1(s,x)p(x)dx$; $Q_2(s,\tau) = G_2(s,\tau) + \omega_2(s)\int_0^1 G_2(\tau,x)q(x)dx$

By (3), we get: $u'(t) = \int_t^1\int_0^1 Q_2(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds - \int_0^1\int_0^1 sQ_2(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds$ (4)

Lemma2.2 [Bai]: Assume (H_1) (H_2) hold. Then one has:

(i) $Q_i(t,s) \ge 0, \forall t,s \in [0,1], Q_i(t,s) > 0, \forall t,s \in (0,1)$;

(ii) $G_i(t,s) \ge a_i G_i(t,t)G_i(s,s), \forall t,s \in [0,1]$;

(iii) $G_i(t,s) \le b_i G_i(s,s), \forall t,s \in [0,1]$.

where $a_1 = 1, a_2 = \sqrt{A}\sin\sqrt{A}; b_1 = 1; b_2 = \dfrac{1}{\sin\sqrt{A}}$.

Denote2: $d_i = \min\limits_{\frac14 \le t \le \frac34} a_i G_i(t,t)(i=1,2)$

Lemma 2.3[Bai]: Assume (H_1) (H_2) hold and are given as above, Then one has

(i) $\max\limits_{0\le t\le 1}\omega_2(t) = \omega_2(\frac12)$; (ii) $0 < d_i < 1$.

Lemma 2.4： If $y(t) \in C[0,1]$ and $y(t) \ge 0$， then the unique solution $u(t)$ of then BVP(1)satisfies:

$\min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t)\ge d_1\|u\|_0$; where: $d_1=\min_{\frac{1}{4}\le t\le\frac{3}{4}} a_1G_1(t,t)$.

Proof: By(2)and (iii) of Lemma2.2,we get:

$$u(t)\le\int_0^1\int_0^1[b_1G_1(s,s)+\omega_1\int_0^1G_1(s,x)p(x)dx]Q_2(s,\tau)y(\tau)d\tau ds$$
$$=\int_0^1\int_0^1[G_1(s,s)+\omega_1\int_0^1G_1(s,x)p(x)dx]Q_2(s,\tau)y(\tau)d\tau ds$$
$$=\int_0^1\int_0^1Q_1(s,s)Q_2(s,\tau)d\tau ds$$

therefore, $\|u\|_0\le\int_0^1\int_0^1Q_1(s,s)Q_2(s,\tau)d\tau ds$.

By (2) and (ii) of Lemma2.2, we have:

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}}u(t)\ge\min_{\frac{1}{4}\le t\le\frac{3}{4}}\int_0^1\int_0^1[a_1G_1(t,t)G_1(s,s)+\omega_1\int_0^1G_1(s,x)p(x)dx]Q_2(s,\tau)y(\tau)d\tau ds$$
$$\ge d_1\int_0^1\int_0^1[G_1(s,s)+\omega_1\int_0^1G_1(s,x)p(x)dx]Q_2(s,\tau)y(\tau)d\tau ds=d_1\int_0^1\int_0^1Q_1(s,s)Q_2(s,\tau)d\tau ds\ge d_1\|u\|_0$$

So, $\min_{\frac{1}{4}\le t\le\frac{3}{4}}u(t)\ge d_1\|u\|_0$, The proof is completed.

Theorem 2.1[guo]: Let $r_2>r_1>0, L>0$ be constants and $\Omega_i=\{u\in X:\alpha(u)<r_i,\beta(u)<L\}, i=1,2$ two bounded open sets in X . Set $D_i=\{u\in X:\alpha(u)=r_i,\}, i=1,2$;

Assume $T:K\to K$ is a completely continuous operator satisfying:

(A_1) $\alpha(Tu)<r_1, u\in D_1\cap K;\alpha(Tu)>r_2, u\in D_2\cap K;$ (A_2) $\beta(Tu)<L, u\in K;$ (A_3)there is $\exists p\in(\Omega_2\cap K)\setminus\{0\}$, such that $\alpha(p)\ne0$ and $\alpha(u+\lambda p)\ge\alpha(u)$,for all $\forall u\in K,\lambda\ge0$.

Then T has at least one fixed point in $(\Omega_2\setminus\overline{\Omega_1})\cap K$.

3.The main results

Let $X=C^1[0,1]$ be the Banach space equipped with the norm $\|u\|=\max_{t\in[0,1]}|u(t)|+\max_{t\in[0,1]}|u'(t)|$, and $K=\left\{u\in X:u\ge0,\min_{t\in[\frac{1}{4},\frac{3}{4}]}u(t)\ge d_1\|u\|_0\right\}$ is a cone in X .

Define functionals $\alpha(u)=\max_{t\in[0,1]}|u(t)|$, $\beta(u)=\max_{t\in[0,1]}|u'(t)|$, $\forall u\in X$, then $\|u\|\le2\max\{\alpha(u),\beta(u)\}$, and $\alpha(\lambda u)=|\lambda|\alpha(u)$, $\beta(\lambda u)=|\lambda|\beta(u)$, $\forall u\in X,\lambda\in R$; $\alpha(u)\le\alpha(v),\forall u,v\in K,u\le v$.

In the following, we denote: $\eta_0=\int_0^1\int_0^1Q_1(s,s)Q_2(s,\tau)d\tau ds$; $\eta_1=\int_0^1\int_{\frac{1}{4}}^{\frac{3}{4}}Q_1(s,s)Q_2(s,\tau)d\tau ds$;

$$\eta_2=\left|\int_0^1[G_2(\tau,\tau)+\omega_2(\tfrac{1}{2})\int_0^1G_2(\tau,x)q(x)dx]d\tau\right|$$

We will suppose that there are $\exists L>b>d_1b>c>0$ such that $f(t,u,v)$ satisfies the following growth conditions:

(H3) $f(t,u,v)<\dfrac{c}{\lambda d_1\eta_0},\forall(t,u,v)\in[0,1]\times[0,c]\times[-L,L]$

(H4) $f(t,u,v)\ge\dfrac{b}{\lambda\eta_1},\forall(t,u,v)\in[0,1]\times[d_1b,b]\times[-L,L]$

(H_5) $f(t,u,v)<\dfrac{2L}{3\lambda b_2\eta_2},\forall(t,u,v)\in[0,1]\times[0,b]\times[-L,L]$

Let $f^*(t,u,v)=\begin{cases}f(t,u,v),(t,u,v)\in[0,1]\times[0,b]\times(-\infty,\infty)\\ f(t,b,v),(t,u,v)\in[0,1]\times(b,\infty)\times(-\infty,\infty)\end{cases}$

and $f_1(t,u,v)=\begin{cases}f^*(t,u,v),(t,u,v)\in[0,1]\times[0,\infty)\times[-L,L]\\ f^*(t,u,-L),(t,u,v)\in[0,1]\times[0,\infty)\times(-\infty,-L]\\ f^*(t,u,L),(t,u,v)\in[0,1]\times[0,\infty)\times[L,\infty)\end{cases}$

Define3:

$$(Tu)(t)=\lambda\int_0^1\int_0^1Q_1(t,s)Q_2(s,\tau)f_1(\tau,u(\tau),u'(\tau))d\tau ds \quad (5)$$

$$(Tu)'(t) = \lambda[\int_t^1\int_0^1 Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds - \int_0^1\int_0^1 sQ_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds] \quad (6)$$

Lemma 3.1: Suppose (H_1) (H_2) hold , then $T: K \to K$ is completely continuous.

Proof: For $u \in K$, by (5) and (iii) of Lemma 2.2, there is $Tu \geq 0$. so

$$\|Tu\|_0 = \max_{t\in[0,1]}\left|\lambda\int_0^1\int_0^1 Q_1(t,s)Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds\right|$$
$$\leq \lambda\int_0^1\int_0^1[b_1G_1(s,s)+\omega_1\int_0^1 G_1(s,x)p(x)dx]Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds$$
$$= \lambda\int_0^1\int_0^1 Q_1(s,s)Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds$$

By Lemma 2.2, (ii), we have :

$$\min_{t\in[\frac14,\frac34]}(Tu)(t) = \min_{t\in[\frac14,\frac34]}\lambda\int_0^1\int_0^1 Q_1(t,s)Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds$$
$$\geq \lambda\int_0^1\int_0^1[a_1G_1(t,t)G_1(s,s)+\omega_1\int_0^1 G_1(s,x)p(x)dx]Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds$$
$$\geq d_1\lambda\int_0^1\int_0^1 Q_1(s,s)Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds = d_1\|Tu\|_0$$

therefore, we get $T(K)\subset K$.

So we can get $T(K)\subset K$ .Let $B\subset K$ is bounded, it is clear that $T(B)$ is bounded. Using $f_1, G_1(t,s), G_2(t,s)$ is continuous, we show that $T(B)$ is equicontinuous. By the Arzela-Ascoli theorem, a standard proof yields $T: K\to K$ is completely continuous.

Theorem 3.1: Suppose condition (H_1)-(H_5)hold, Then BVP(1)has at least one positive solution $u(t)$ satisfying: $c<\alpha(u)<b, \beta(u)<L$.

Proof : Take $\Omega_1 = \{u\in X: |u(t)|<c, |u'(t)|<L\}$, $\Omega_2 = \{u\in X: |u(t)|<b, |u'(t)|<L\}$ two bounded open sets in X and $D_1 = \{u\in X: \alpha(u)=c\}$, $D_2=\{u\in X:\alpha(u)=b\}$.

By Lemma3.1, $T:K\to K$ is completely continuous operator, and there is $\exists p\in(\Omega_2\cap K)\setminus\{0\}$, such that $\alpha(p)\neq 0$ and $\alpha(u+\lambda p)\geq\alpha(u), \forall u\in K, \lambda\geq 0$. $\forall u\in D_1\cap K, \alpha(u)=c$.

From(H_3)we get:

$$\alpha(Tu) = \max_{t\in[0,1]}\left|\lambda\int_0^1\int_0^1 Q_1(t,s)Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds\right|$$
$$< \lambda\times\frac{c}{\lambda d_1\eta_0}\max_{t\in[0,1]}\left|\int_0^1\int_0^1[a_1G_1(t,t)G_1(s,s)+\omega_1\int_0^1 G_1(s,x)p(x)dx]Q_2(s,\tau)d\tau ds\right|$$
$$< d_1\times\frac{c}{d_1\eta_0}\int_0^1\int_0^1[G_1(s,s)+\omega_1\int_0^1 G_1(s,x)p(x)dx]Q_2(s,\tau)d\tau ds$$
$$= \frac{c}{\eta_0}\int_0^1\int_0^1 Q_1(s,s)Q_2(s,\tau)d\tau ds = c$$

Where as for $\forall u\in D_2\cap K, \alpha(u)=b$ .From Lemma 2.4,we have $u(t)\geq d_1\alpha(u)=d_1b$, $t\in[\frac14,\frac34]$ .so from (H_4) we get:

$$\alpha(Tu) = \max_{t\in[0,1]}\left|\lambda\int_0^1\int_0^1 Q_1(t,s)Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds\right|$$
$$> \lambda\times\frac{b}{\lambda\eta_1}\max_{t\in[0,1]}1\left|\int_0^1\int_{\frac14}^{\frac34}[b_1G_1(s,s)+\omega_1\int_0^1 G_1(s,x)p(x)dx]Q_2(s,\tau)d\tau ds\right|$$
$$= \frac{b}{\eta_1}\int_0^1\int_{\frac14}^{\frac34} Q_1(s,s)Q_2(s,\tau)d\tau ds = b$$

$\forall u\in K$, From (H_5) we get:

$$\beta(Tu) = \max_{t\in[0,1]}\left|\lambda\int_t^1\int_0^1 Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds - \lambda\int_0^1\int_0^1 sQ_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))d\tau ds\right|$$
$$< \lambda\times\frac{2L}{3\lambda b_2\eta_2}\max_{t\in[0,1]}\left|\int_0^1\int_0^1(1+s)Q_2(s,\tau)d\tau ds\right|$$
$$< \frac{2L}{3b_2\eta_2}\left|\int_0^1\int_0^1(1+s)[b_2G_2(\tau,\tau)+\max_{s\in[0,1]}\omega_2(s)\int_0^1 G_2(\tau,x)q(x)dx]d\tau ds\right|$$

$$< \frac{2L}{3b_2\eta_2} \times \frac{3}{2} b_2 \left| \int_0^1 [G_2(\tau,\tau) + \omega_2(\tfrac{1}{2}) \int_0^1 G_2(\tau,x) q(x) dx] d\tau \right| = L$$

Theorem 2.1 implies there is $u \in (\Omega_2 \setminus \bar{\Omega}_1) \cap K$, such that $u = Tu$ .so, $u(t)$ is a positive solution for BVP(1), satisfying : $c < \alpha(u) < b, \max_{t\in[0,1]} |u^{'}(t)| < L$.

Thus, Theorem 3.1is completed.

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