H_∞ Filter for Continuous-time Systems With Measurement Losses

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Abstract: The H_∞ filter for continuous-time linear systems with measurement losses is studied by using lose ratio. The optimal strategies of the filter are provided and its optimality is proved. Then, the necessary and sufficient conditions of the optimal strategies are solved. Finally, the effectiveness of the filter is verified.

Introduction

With the computer technology and network communication technology introduced into control system, the control system structure become more and more complex. The emergence of the H_∞ filter method makes up the defect of the modern control theory. The assumption of noise can be neglected in H_∞ filter, then it's more accurate description for real system noise[1]. There is absolute advantage for H_∞ filter when the requirement of every state is different. Due to the influence of various factors, packet dropouts and time-delay is a common phenomenon in network communication.

A class of networked control systems with uncertain parameters and time-delay and packet losses in the network communication are studied in [2-9]. variety H_∞ filters with time-delay saturation system are designed in [1]. H_∞ filter for discrete time switched systems with measurement losses are analyzed in [10]. H_∞ filter for discrete-time systems with measurement losses has been analyzed deeply, but similar problem couldn't be studied for continuous-time systems. Since the continuous system is common in reality, H_∞ filter for continuous-time with measurement losses is analyzed in this paper.In this paper, the continuous-time system with random measurement losses is analyzed. The stable filter has been derived and the optimal condition for the stability of the filter is proved.

Problem statement

A continuous-time linear system with measurement losses described by

$$\begin{cases} \dot{x} = Fx + \omega \\ y = \gamma Cx + v \\ z = Lx \end{cases} \tag{1}$$

Where x is the $n\times 1$ state vector y is the $m\times 1$ measurement vector; ω and v are the noise, F, C and L are time-varying matrices of appropriate dimensions. γ are Bernoulli distributed random variables. If $\gamma=1$, the measurement is received. If $\gamma=0$,the measurement is lost. Z is a linear combination of the states x ,where L is $q\times m$ matrix, when L=I, the vectors reduces to the state vector. And Z is the aim of optimal estimation.

the cost function is defined in the filtering method based on game theory.

$$J_1 = \frac{\int_0^T \|z-\hat{z}\|_S^2 \, dt}{\|x_0-\hat{x}_0\|_{P_0^{-1}}^2 + \int_0^T \left(\|\omega_k\|_{Q^{-1}}^2 + \|v_k\|_{R^{-1}}^2\right) dt} \tag{2}$$

Where P_0, Q, R, S are symmetric matrices of appropriate dimensions. It is difficult to compute the minimum J_1 directly. First, performance boundaries makes $J_1 < 1/\theta$. θ is the specified performance boundaries. Then, we know

$$J = \int_0^T \left[\|z - \hat{z}\|_S^2 - \frac{1}{\theta}\left(\|\omega\|_{Q^{-1}}^2 + \|v\|_{R^{-1}}^2 \right) \right] dt - \frac{1}{\theta}\|x_0 - \hat{x}_0\|_{P_0^{-1}}^2 < 0 \tag{3}$$

Thus, the problem is converted to solve the minimax problems, that is

$$J^* = \min_{\hat{z}} \max_{\omega, y, x_0} J \tag{4}$$

It can be further derived by the relationship $\hat{z} = L\hat{x}$ and $y = \gamma Hx + v$.

$$J = \int_0^T \left[\|x - \hat{x}\|_{\bar{S}}^2 - \frac{1}{\theta}\left(\|\omega\|_{Q^{-1}}^2 + \|y - \gamma Cx\|_{R^{-1}}^2 \right) \right] dt - \frac{1}{\theta}\|x_0 - \hat{x}_0\|_{P_0^{-1}}^2 \tag{5}$$

Where $\bar{S} = L^T S\, L$, in order to solve J^*, First, the extremum of J on x_0 and w must be solved; then, the extremum on $\hat{x}$ and y could be solved.

H_∞ filter analysis

Assuming a continuous-time system:

$$\begin{cases} \dot{x} = Fx + \omega \\ y = \gamma Cx + v \\ z = Lx \end{cases} \tag{6}$$

Then, the system is controlled and detected and γ is known at any time.

A. The extremum of J on w and x_0

In this section, the maximum of cost function J on w and x_0 is solved under the constraints of $\dot{x} = Fx + \omega$. It can be considered as a dynamic constrained optimal problem. First, the Hamilton function can be defined, resulting in

$$H = \frac{1}{2}\left(\|x - \hat{x}\|_{\bar{S}}^2 - \frac{1}{\theta}\left(\|\omega\|_{Q^{-1}}^2 + \|y - \gamma Cx\|_{R^{-1}}^2 \right) \right) + \frac{\lambda^T}{\theta}(Fx + \omega) \tag{7}$$

Where, the $\dfrac{\lambda^T}{\theta}$ is time-varying Lagrange multiplier. the necessary condition for the maximum is: $x(0) = \hat{x}(0) + P_0\lambda(0), \lambda(T) = 0$ $\dot{\lambda} = -A^T\lambda - \theta\bar{S}(x - \hat{x}) - \gamma C^T R^{-1}(y - \gamma Cx)$.

Then:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & Q \\ \gamma C^T R^{-1} C - \theta\bar{S} & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 \\ \theta\bar{S}\hat{x} - \gamma C R^{-1} y \end{pmatrix} \tag{8}$$

From $x_0 = \hat{x}_0 + P_0\lambda_0$, we can assume $x^* = x_P + P\lambda^*$ .where x_p and P are unknown function x^* are λ^* homologous optimal estimations.

The optimal estimation of w and x_0 are:

$$\omega^* = Q\lambda^*, x(0) = \hat{x}_0 + P_0\lambda(0)^* \tag{9}$$

Differentiating (9) and substituting for x^* are λ^* from (8) results in

$$\dot{x}_p - Ax_p - PC^T R^{-1}(y - \gamma Cx_p) + \theta R\bar{S}(x_p - \hat{x}) = \left[AP + PA^T + Q - P\left(C^T R^{-1} C - \theta\bar{S}\right)P - \dot{P} \right]\lambda^* \tag{10}$$

For the equation (9), the left side and the right side are set identically to zero, resulting in

$$\dot{x}_p = Ax_p + \gamma PC\bar{S}(y - \gamma Cx) - \theta P\bar{S}(x_p - \hat{x}) \quad x_p(0) = \hat{x}(0) \quad \dot{P} = AP + PA^T + Q - P\left(C^T R^{-1} C - \theta\bar{S}\right)P \quad P(0) = P_0 \tag{11}$$

Where (11) is Riccati differential equation, $P(t)$ is solution of equation.

B. The extremum of J on $\hat{x}$ and y

In this section, the mainly aim is computing the extremum of J on $\hat{x}_k$ and y_k based on last section's solve. From last section, we can know the optimal strategy, resulting in

$\omega^* = Q\lambda^*, x(0) = \hat{x}_0 + P_0\lambda(0)^*$

From (8) are substituted into the cost function, and adding the identically zero term

$$\frac{1}{2\theta}\left\|\lambda^*(0)\right\|_{P_0}^2 - \frac{1}{2\theta}\left\|\lambda^*(T)\right\|_{P(T)}^2 + \frac{1}{2\theta}\int_0^T d/dt\left(\left\|\lambda^*(T)\right\|_{P(T)}^2\right)dt = 0 \quad (12)$$

Results in the min-max problem

$$\min_{\hat{x},y}\max J = \frac{1}{2}\int_0^T (\left\|x-\hat{x}\right\|_{\bar{S}}^2 - \frac{1}{\theta}\left(\left\|\omega\right\|_{Q^{-1}}^2 + \left\|y-\gamma Cx\right\|_{R^{-1}}^2\right))dt \quad (13)$$

Subject to the dynamic constraints

$$\dot{x}_p = Ax_p + \gamma PC\bar{S}(y-\gamma Cx) - \theta P\bar{S}(x_p - \hat{x}) \quad x_p(0) = \hat{x}(0) \; \dot{P} = AP + PA^T + Q - P\left(\gamma C^T R^{-1} C - \theta\bar{S}\right)P \quad P(0) = P_0 \quad (14)$$

Defining variables are $r = x_p - \hat{x}, q = y - \gamma Cx_p$. Because r is independent with q, $r^* = 0, q^* = 0$ can be derived by (12-13).

From (9) and (13), we can know when $\hat{x}^* = x_p, \hat{y}^* = \gamma Cx_p$, the optimal strategy is derived, resulting in

$$\hat{x}^* = x_p, \hat{y}^* = \gamma Cx_p \quad \omega^* = Q\lambda^*, x(0) = \hat{x}_0 + P_0\lambda(0)^* \quad x^* = x_p + P\lambda^* \quad (15)$$

Then the value of cost function can result in $J\left(\hat{x}^*, y^*, \omega^*, x(0)^*\right) = 0$.

C. Optimal condition

In this section, we can prove that the existence of $P(t)$ is a necessary and sufficient condition for optimal strategy. For the optimal strategy, there must be $\Delta J = J\left(\hat{x}^*, y, \omega, x(0)\right) - J\left(\hat{x}^*, y^*, \omega^*, x(0)^*\right) \le 0$.

Sufficiency: Assume that $P(t)$ is bounded, the optimal strategies satisfy a saddle-point inequality. That is

$$J\left(\hat{x}^*, y, \omega, x(0)\right) \le J\left(\hat{x}^*, y^*, \omega^*, x(0)^*\right) \le J\left(\hat{x}, y^*, \omega^*, x(0)^*\right) \quad (16)$$

Where $\hat{x}$ 、 y 、 ω 、 $x(0)$ show any strategies.

Necessity: Assume that $P(t)$ is unbounded, resulting in

$$\Delta J = \lim_{\varepsilon\to 0}(-\frac{1}{2\theta}\left\|\Delta e^*(t_1-\varepsilon)\right\|_{p^{-1}(t_1-\varepsilon)}^2 - \frac{1}{2\theta}\int_0^{(t_1-\varepsilon)}\left\|QP^{-1}(\Delta e^*) - \omega\right\|_{Q^{-1}}^2 + \left\|y-\gamma C\hat{x}^*\right\|_{R^{-1}}^2 dt) + \frac{1}{2}\int_{(t_1-\varepsilon)}^T \left\|\Delta e^*\right\|_{\bar{S}}^2 - \frac{1}{\theta}\left(\left\|\omega\right\|_{Q^{-1}}^2 + \left\|y-\gamma Cx\right\|_{\bar{S}}^2\right)dt \quad (17)$$

Where $\Delta e^* = x - \hat{x}^*$ represents error in the optimal estimate. t_1 is any time and is a little value.

Due to the existence of $P(t)$ is unknown. So the dynamic constraints don't work in section A. Considering this strategy

$$\omega = QP^{-1}\Delta e^*, y = \gamma C\hat{x}^* \quad \forall t \in [0, t_1 - \varepsilon] \quad (18)$$

$$\omega = 0, y = \gamma Cx \quad \forall t \in [t_1 - \varepsilon, T] \quad (19)$$

From (18-19) are substituted into ΔJ, resulting in

$\Delta J = \lim_{\varepsilon\to 0} -\frac{1}{2\theta}\left\|\Delta e^*(t_1-\varepsilon)\right\|_{P_{(t_1-\varepsilon)}^{-1}}^2 + \frac{1}{2}\int_{(t_1-\varepsilon)}^T \left\|\Delta e^*\right\|_{\bar{S}}^2 dt$ with $\varepsilon \to 0$, $P^{-1}(t_1-\varepsilon)$ tends to a singular matrix. The former can be shown as $\Delta e^*(t_1-\varepsilon) = \Phi(t_1-\varepsilon, 0)\Delta e^*(0)$.

Where $\Phi(\bullet) = A + QP^{-1}$ is state transition matrix and $\Delta e^*(0) = x(0) - \hat{x}_0$.

Since $x(0)$ is artrary, $x(0)$ can be chosen as $\lim_{\varepsilon\to 0}\Delta e^*(t_1-\varepsilon)$, so

$$\lim_{\varepsilon\to 0} -\frac{1}{2\theta}\left\|\Delta e^*(t_1-\varepsilon)\right\|_{p^{-1}(t_1-\varepsilon)}^2 = 0 \quad (20)$$

$$\Delta J = \lim_{\varepsilon\to 0}\frac{1}{2}\int_{(t_1-\varepsilon)}^T \left\|\Delta e^*\right\|_{\bar{S}}^2 dt > 0 \quad (21)$$

Therefore, (21) is contradiction with assumption. Hence, $P(t)$ is bounded at [0,T].

Simulation examples

Consider the following scalar continuous system:

$$\begin{cases} \dot{x} = x + \omega \\ y = \gamma x + v \\ z = x \end{cases} \quad (22)$$

So A=C=L=1. Assuming that Q=R=S=1, the Riccati differential equation can be derived.

$$\dot{P} = AP + PA^T + Q - \gamma PC^T R^{-1} CP + \theta PL^T SLP = 2P + 1 + (\theta - \gamma) P^2 \quad (23)$$

Where loss probability is 0.2 and the value of θ is 7/16. The simulation results are as follows.

This is the estimation error curve. The MSE curve of all measurement received is green, however, when the loss probability of measurement is 0.2, the MSE curve is red. From this figure, we can know the filter is convergence effectively, but the performance is falling.

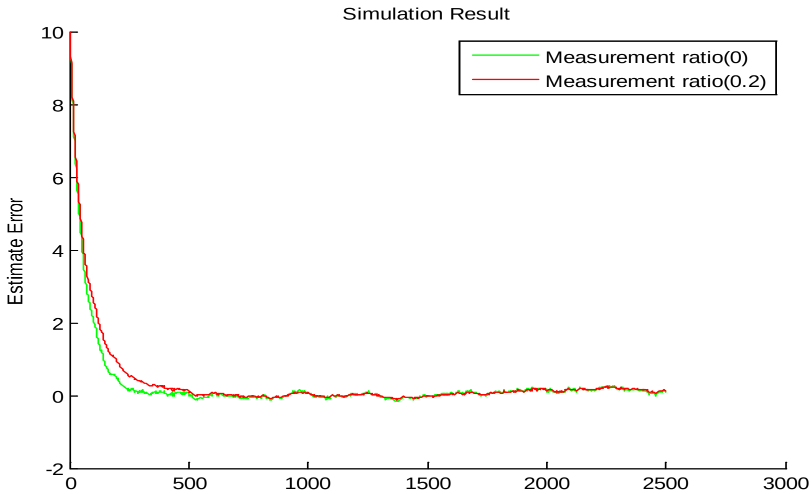

Figure 1 Simulation Result

Conclusion

In this paper, continuous-time H∞ filter with random measurement losses is designed for the foundation problem of measurement losses in networked control systems. The structure of the filter is described in section 2. The optimal filter is solved and the necessary and sufficient conditions of the optimal filter is that $P(t)$ is bounded. Finally, the effectiveness of filter is validated by simulation.

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