Some inequalities for the generalized trigonometric and hyperbolic functions

Bao-ju Sun
Department of Mathematics,
Zhejiang University of Water Resources and Electric Power,
Hangzhou, Zhejiang 310018, China
sunbj@zjweu.edu.cn

***Abstract*—Some classical inequalities for trigonometric and hyperbolic functions, such as Shafer-Fink inequality, Shafer-Fink type inequality, and Wilker-type inequality, are generalized to the case of generalized functions. A conjecture posed by Klen, Vuorinen and Zhang [Journal of Mathematical Analysis and Applications, 409 (2014) 521-529] is proved.**

***Keywords*— generalized trigonometric functions; generalized hyperbolic functions; Shafer-Fink inequality; Wilker-type inequality**

I. INTRODUCTION

It is well known from basic calculus that

$$\arcsin(x)=\int_0^x \frac{1}{(1-t^2)^{1/2}}\mathrm{d}t,\quad 0\le x\le 1,$$

and

$$\frac{\pi}{2}=\arcsin(1)=\int_0^1 \frac{1}{(1-t^2)^{1/2}}\mathrm{d}t.$$

For $1<p<\infty$, We can generalize the above function as follows:

$$\arcsin_p(x)\equiv\int_0^x \frac{1}{(1-t^p)^{1/p}}\mathrm{d}t,\quad 0\le x\le 1,$$

and

$$\frac{\pi_p}{2}=\arcsin_p(1)=\int_0^1 \frac{1}{(1-t^p)^{1/p}}\mathrm{d}t.$$

The inverse of $\arcsin_p$ on $[0,\pi_p/2]$ is called the generalized sine function and denoted by $\sin_p$.

The generalized cosine function $\cos_p$ is defined as

$$\cos_p(x)\equiv\frac{\mathrm{d}}{\mathrm{d}x}\sin_p(x).$$

It is clear from the definition that

$$\cos_p(x)=\left(1-\sin_p(x)^p\right)^{1/p}.$$

The generalized tangent function $\tan_p$ is defined as

$$\tan_p(x)\equiv\frac{\sin_p(x)}{\cos_p(x)}.$$

Similarly, the generalized inverse hyperbolic sine function $\operatorname{arcsinh}_p$ is defined as

$$\operatorname{arcsinh}_p(x)=\int_0^x \frac{1}{(1+t^p)^{1/p}}\mathrm{d}t,\quad x\ge 0.$$

The inverse of $\operatorname{arcsinh}_p$ on $[0,\infty]$ is called the generalized hyperbolic sine function and denoted by $\sinh_p$. The generalized hyperbolic cosine function $\cosh_p$ is defined as

$$\cosh_p(x)\equiv\frac{\mathrm{d}}{\mathrm{d}x}\sinh_p(x).$$

It is clear from the definition that

$$\cosh_p(x)=\left(1+\sinh_p(x)^p\right)^{1/p}.$$

The generalized hyperbolic tangent function $\tanh_p$ is defined as

$$\tanh_p(x)\equiv\frac{\sinh_p(x)}{\cosh_p(x)}.$$

when $p=2$, the $p-$ functions $\sin_p$, $\cos_p$, $\tan_p$, $\sinh_p$, $\cosh_p$, $\tanh_p$ become our familiar trigonometric and hyperbolic functions.

Recently, the generalized trigonometric and hyperbolic functions have been studied by many mathematicians from differeent viewpoints(see[1-7]). In [1], the authors gave basic properties of the generalized trigonometric functions. In [4], Kl é n, Vuorinen and Zhang generalized some classical inequalities for trigonometric and hyperbolic functions, such as Mitrinović-Adamović's inequality, Lazarević's inequality, Huygens-type inequalities, and Wilker-type inequalities, to the case of generalized functions.

Theorem 1 (Mitrinović-Adamović's inequalities [4, p524 Theorem 3.6] **)** *For* $p>1$, *the following inequalities hold*

$$\cos_p(x)^{1/(1+p)}<\frac{\sin_p(x)}{x}<1,\ 0<x<\pi_p/2. \tag{1}$$

Theorem 2 (Wilker-type inequality[4, p525 Corollary 3.19] **)** *For* $p>1$, *the following inequality hold*

$$\left(\frac{\sinh_p(x)}{x}\right)^2+\frac{\tanh_p(x)}{x}>2, \quad x>0. \qquad (2)$$

Morecover, Klén, Vuorinen and Zhang raised the conjecture.

Conjecture 3 ([4, p527 conjecture 3.29]**)** *For* $p>2,\ 0<x<\pi_p/2$. *the following inequality*

$$\frac{\sinh_p(x)}{x}<\frac{p+1}{p+\cos_p(x)} \qquad (3)$$

hold.

In [7], the author gave Shafer-Fink inequality for the generalized trigonometric functions.

Theorem 4 *For* $1<p\le 2$, *the following inequalities.*

$$\frac{(1+p)x}{p+(1-x^p)^{1/p}}\le \arcsin_p(x)\le \frac{p\pi_p x}{2(p+(1-x^p)^{1/p})} \qquad (4)$$

holds for all $0\le x\le \pi_p/2$.

In this paper we will give Shafer-Fink inequality for the generalized hyperbolic functions and Shafer-Fink type inequalities for the generalized trigonometric and hyperbolic functions. we also give Wilker-type inequality for the generalized trigonometric functions. Moreover we verify a conjecture posed by Klén, Vuorinen and Zhang .

II. LEMMAS

The following lemmas are needed in sequel.

Lemma 5 (L'Hopital Monotone Rule see [8])

let $-\infty<a<b<\infty$, *and let* $f,g:[a,b]\to\mathbb{R}$ *be continuous functions that are differentiable on* (a,b) , *with* $f(a)=g(a)=0$ *or* $f(b)=g(b)=0$ *Assume that* $g'(x)\ne 0$ *for each* $x\in(a,b)$. *If* f'/g' *is increasing (decreasing) on* (a,b), *then so is* f/g .

Lemma 6 *For* $p>2$, *let* $0<u<1$, *then*

$$H(u)=2u^{2-p}-p+(p-1)u^{3-p}-u>0. \qquad (5)$$

Proof. For $p\ge 3$,

$$H'(u)=2(2-p)u^{1-p}+(p-1)(3-p)u^{2-p}-1<0,$$

Which implies

$$H(u)>H(1)=0.$$

For $2<p<3$,

$$H''(u)=2(2-p)(1-p)u^{-p}+(p-1)(3-p)(2-p)u^{1-p}$$
$$=(2-p)(1-p)u^{-p}[2-(3-p)u]>0.$$

We have

$$H'(u)<H'(1)=2p-p^2<0,$$

This implies

$$H(u)>H(1)=0.$$

Therefor inequality (5) holds.

III. RESULTS AND PROOFS

The first result and its proof procedure are presented as follows.

Theorem 7 *For* $1<p\le 2$, *let* $0\le x\le r$, *and* $r>0$, *the following inequalities.*

$$\frac{(1+p)x}{p+(1+x^p)^{1/p}}\le \operatorname{arcsinh}_p(x)\le \frac{bx}{p+(1+x^p)^{1/p}} \qquad (6)$$

holds, where $b=\dfrac{\left(\operatorname{arcsinh}_p(r)\right)\left(p+(1+r^p)^{1/p}\right)}{r}$.

Proof. Let $\operatorname{arcsinh}_p(x)=t$, and

$$f(t)=\frac{t(p+\cosh_p(t))}{\sinh_p(t)}.$$

By differentiation, we have

$$f'(t)=\frac{1}{\sinh_p(t)^2}g(t),$$

$$g(t)=p\sinh_p(t)+\sinh_p(t)\cosh_p(t)$$
$$-pt\cosh_p(t)-t\cosh_p(t)^{2-p}.$$

A simple computation leads to

$$g'(t)=\cosh_p(t)^{2-p}[2\sinh_p(t)^p-pt\sinh_p(t)^{p-1}$$
$$-(2-p)t\cosh_p(t)^{1-p}\sinh_p(t)^{p-1}]$$
$$=\cosh_p(t)^{2-p}\sinh_p(t)^{p-1}$$
$$\times[2\sinh_p(t)-pt-(2-p)t\cosh_p(t)^{1-p}]$$
$$=\cosh_p(t)^{2-p}\sinh_p(t)^{p-1}h(t),$$

where

$$h(t)=2\sinh_p(t)-pt-(2-p)t\cosh_p(t)^{1-p},$$

and

$$h'(t)=2\cosh_p(t)-p-(2-p)\cosh_p(t)^{1-p}$$
$$-(2-p)(1-p)\cosh_p(t)^{2-2p}\sinh_p(t)^{p-1}$$
$$>(2-p)(\cosh_p(t)-\cosh_p(t)^{1-p})$$
$$+(2-p)(p-1)\cosh_p(t)^{2-2p}\sinh_p(t)^{p-1}\ge 0.$$

Now it is easy to see that $f'(t)>0$, therefor

$$p+1=f(+0)<f(t)$$

$$\leq f(\operatorname{arcsinh}_p(r))=\frac{\left(\operatorname{arcsin}h_p(r)\right)\left(p+(1+r^p)^{1/p}\right)}{r}=b.$$

This implies the inequality (6). □

The inequality (6) is so-called Shafer-Fink inequality[12,13.14]. The following Theorem 8 and 9 present the Shafer-Fink type inequalities[12,13,14].

Theorem 8 *For* $1<p\leq 2$, *let* $0\leq x\leq 1$, *the following inequalities*

$$\frac{(1+p)x}{p+(1-x^p)^{1/p}}\leq \arcsin_p(x)\leq \frac{\pi_p x}{2+(\pi_p-2)(1-x^p)^{1/p}} \tag{7}$$

holds.

Proof. Let $\arcsin_p(x)=t,$ and

$$f(t)=\frac{\sin_p(t)-t\cos_p(t)}{t-\sin_p(t)}.$$

Write

$f_1(t)\equiv \sin_p(t)-t\cos_p(t),$ and $f_2(t)\equiv t-\sin_p(t),$

then $f_1(0)=0,\ f_2(0)=0,$ by simple computations,

$$\frac{f_1'(t)}{f_2'(t)}=\frac{t\cos_p(t)^{2-p}\sin_p(t)^{p-1}}{1-\cos_p(t)}=\frac{f_{11}(t)}{f_{22}(t)},$$

with $f_{11}(t)=t\cos_p(t)^{2-p}\sin_p(t)^{p-1}$,

$f_{22}(t)=1-\cos_p(t),\ f_{11}(0)=0,\ f_{22}(0)=0,$

$$\frac{f_{11}'(t)}{f_{22}'(t)}=1+\left[(p-2)t\frac{\sin_p(t)^{p-1}}{\cos_p(t)^{p-1}}+(p-1)t\frac{\cos_p(t)}{\sin_p(t)}\right],$$

Write $\dfrac{g_1(t)}{g_2(t)}=\dfrac{t\cos_p(t)}{\sin_p(t)}$,

$$\frac{g_1'(t)}{g_2'(t)}=1-t\tan_p(t)^{p-1}$$

Which is strictly decreasing and hence so is $\dfrac{f_{11}'(t)}{f_{22}'(t)}$, by the L'Hopital Monotone Rule we see that $f(t)$ is strictly decreasing, leads to

$$\frac{2}{\pi_p-2}=f(\frac{\pi_p}{2})\leq f(t)=\frac{\sin_p(t)-t\cos_p(t)}{t-\sin_p(t)}<f(+0)=p.$$

This implies the inequality(7). □

Theorem 9 *For* $1<p\leq 2$, *let* $0\leq x\leq r$, *and* $r>0$, *the following inequalities*

$$\frac{(1+p)x}{p+(1+x^p)^{1/p}}\leq \operatorname{arcsin}_p(x)\leq \frac{(b+1)x}{b+(1+x^p)^{1/p}} \tag{8}$$

holds.

where $b=\dfrac{(1+r^p)^{1/p}\left(\operatorname{arcsin}h_p(r)\right)-r}{r-\operatorname{arcsin}h_p(r)}.$

Proof. Let $\operatorname{arcsin}h_p(x)=t,$ and

$$F(t)=\frac{t\cosh_p(t)-\sinh_p(t)}{\sinh_p(t)-t}=\frac{A(t)}{B(t)}$$

Write

$A(t)\equiv t\cosh_p(t)-\sinh_p(t),\ B(t)\equiv \sinh_p(t)-t,$

then $A(0)=0,\ B(0)=0,$ by simple computations,

$$\frac{A'(t)}{B'(t)}=\frac{t\cosh_p(t)^{2-p}\sinh_p(t)^{p-1}}{\cosh_p(t)-1}=\frac{A_1(t)}{B_2(t)},$$

with $A_1(t)=t\cosh_p(t)^{2-p}\sinh_p(t)^{p-1}$,

$B_1(t)=\cosh_p(t)-1,$ and $A_1(0)=0,\ B_1(0)=0,$

$$\frac{A_1'(t)}{B_1'(t)}=1+\left[(2-p)t\frac{\sinh_p(t)^{p-1}}{\cosh_p(t)^{p-1}}+(p-1)t\frac{\cosh_p(t)}{\sinh_p(t)}\right]$$

$$=1+\left[(2-p)t\tanh_p(t)^{p-1}+(p-1)\frac{t\cosh_p(t)}{\sinh_p(t)}\right],$$ Write

$$\frac{g_1(t)}{g_2(t)}=\frac{t\cosh_p(t)}{\sinh_p(t)}$$

$$\frac{g_1'(t)}{g_2'(t)}=1+t\tanh_p(t)^{p-1}$$

Which is strictly increasing and hence so is $\dfrac{A_1'(t)}{B_1'(t)}$, by the L'Hopital Monotone Rule we see that $F(t)$ is strictly increasing, leads to

$$p=F(+0)<F(t)$$

$$\leq F(\operatorname{arcsinh}_p(r))=\frac{(1+r^p)^{1/p}\left(\operatorname{arcsin}h_p(r)\right)-r}{r-\operatorname{arcsinh}_p(r)}=b.$$

This implies the inequality (8). □

The next theorem show the Wilker_type inequality for generalized trigonometric functions[10].

Theorem10 *For* $p>1$, *let* $0<x<\pi_p/2$, *the following inequality holds.*

$$\left(\frac{\sin_p(x)}{x}\right)^p+\frac{\tan_p(x)}{x}>2.$$

Proof. The first inequality in (1) can be re-written as

$$\left(\frac{\sin_p(x)}{x}\right)^p\frac{\tan_p(x)}{x}>1$$

So

$$\left(\frac{\sin_p(x)}{x}\right)^p+\frac{\tan_p(x)}{x}\geq 2\sqrt{\left(\frac{\sin_p(x)}{x}\right)^p\frac{\tan_p(x)}{x}}>2.$$

□

The next theorem answer the conjecture affirmatively.

Theorem11 F*or* $p>2$, $0<x<\pi_p/2$. *the following inequality*

$$\frac{\sinh_p(x)}{x}<\frac{p+1}{p+\cos_p(x)}$$

hold.

Proof. Let $f(x)=(p+1)x-p\sinh_p(x)-\sinh_p(x)\cos_p(x)$,

By simple computations, we give

$$f'(x)=p+1-p\cosh_p(x)-\cosh_p(x)\cos_p(x)$$
$$+\sinh_p(x)\sin_p(x)^{p-1}\cos_p(x)^{2-p},$$

and

$$f''(x)=-p\cosh_p(x)\tanh_p(x)^{p-1}-\sinh_p(x)\cos_p(x)\tanh_p(x)^{p-2}$$
$$+2\cosh_p(x)\sin_p(x)^{p-1}\cos_p(x)^{2-p}$$
$$+(p-1)\sinh_p(x)\cos_p(x)^{3-p}\sin_p(x)^{p-2}$$
$$+(p-2)\sinh_p(x)\cos_p(x)^{3-2p}\sin_p(x)^{2p-2}$$
$$\geq 2\cosh_p(x)\sin_p(x)^{p-1}\cos_p(x)^{2-p}-p\cosh_p(x)\tanh_p(x)^{p-1}$$
$$+(p-1)\sinh_p(x)\cos_p(x)^{3-p}\sin_p(x)^{p-2}$$
$$-\sinh_p(x)\cos_p(x)\tanh_p(x)^{p-2}$$
$$=2\cosh_p(x)\sin_p(x)\sin_p(x)^{p-2}\cos_p(x)^{2-p}-p\sinh_p(x)\tanh_p(x)^{p-2}$$
$$+(p-1)\sinh_p(x)\cos_p(x)^{3-p}\sin_p(x)^{p-2}$$
$$-\sinh_p(x)\cos_p(x)\tanh_p(x)^{p-2}$$
$$\geq 2\sinh_p(x)\sin_p(x)^{p-2}\cos_p(x)^{2-p}-p\sinh_p(x)\tanh_p(x)^{p-2}$$
$$+(p-1)\sinh_p(x)\cos_p(x)^{3-p}\sin_p(x)^{p-2}$$
$$-\sinh_p(x)\cos_p(x)\tanh_p(x)^{p-2}$$
$$\geq\sinh_p(x)\tanh_p(x)^{p-2}G(x), \quad (9)$$

with

$$G(x)=\left[2\cos_p(x)^{2-p}-p+(p-1)\cos_p(x)^{3-p}-\cos_p(x)\right]$$

the inequality (9) follows from

$$\sin_p(x)>\tanh_p(x)\quad (or\cosh_p(x)\sin_p(x)>\sinh_p(x)).$$

By lamma 6 we get $f''(x)>0,$ so $f'(x)>f'(0)=0,$

This implies $f(x)>f(0)=0$.

The theorem is proved. □

REFERENCES

[1] D. E. Edmundes, P. Gurka, and J. Lang, “Properties of generalized trigonometric functions,” *Journal of ApproximationTheory*,vol. 164, no.1, pp. 47-56, 2012.

[2] W. –D. Jiang, M. –K. Wang, Y. –M. Chu, Y. –P.Jiang, and F. Qi, “Converxity of the generalized sine function and the generalized hyperbolic sine function,” J *Journal of ApproximationTheory*,vol. 174, no.1, pp. 1-9, 2013

[3] B. A. Bhayo, and M. Vuorinen, “On generalized trigonometric functions with two parameters,” *Journal of ApproximationTheory* , vol.164, no. 10, pp. 1415-1426, 2012.

[4] R. Klén, M. Vuorinen, and X. –H. Zhang, “Inequalities for the generalized trigonometric and hyperbolic functions,” *Journal of Mathematical Analysis and Applications*, vol. 409, no.1, pp. 521-529, 2014.

[5] Y. –Q. Song, Y. –M. Chu, B. –Y. Liu, and M. –K. Wang, “A note on generalized trigonometric and hyperbolic functions,” *Journal of Mathematical inequalities*, 2014, in press.

[6] S. Takeuchi, “Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian,” *Journal of Mathematical Analysis and Applications*, vol. 385, no. 1, pp. 24-35, 2012.

[7] C. –Y. Yang, “Inequalities on generalized trigonometric and hyperbolic functions,” *Journal of Mathematical Analysis and Applications*, vol. 419, no.1, pp. 775-782, 2014.

[8] G. D. Anderson,M.K.Vamanamurthy, and M.Vuorinen, “Monotonicity rules in calculus,” *The American Mathematical Monthly*, vol.133, no.1, pp. 805-816, 2006.

[9] G. D. Anderson, M.Vuorinen, and X.-H. Zhang , *Topics on special functions III.* arXiv:1209.1696v1[math.CA].

[10]] J. B. Wilker, Problem E 3306, T *The American Mathematical Monthly*, vol. 96, no.1, pp. 55, 1989.

[11]] A. Baricz and J. Sandor, “Extensions of generalized Wilker inequality to Bessel functions,” *Journal of Mathematical inequalities*, vol.2, no.3, pp. 397-406, 2008.

[12] B. J. Malesević, “An application of $\lambda-$ method on inequalities of Shafer-Fink’s type," *Mathematical inequalities and Applications*, vol. 10, no.3, pp. 529-534, 2007.

[13] L.Zhu, “On Shafer-Fink’s type inequalities,” *Journal of inequalities and Applications*, vol. 2007, Artice ID 67430,4 pages, 2007.

[14] L.Zhu, “New inequalities of Shafer-Fink type for arc-hyperbolic sine,” *Journal of inequalities and Applications*, vol. 2008, Artice ID 368275, 5 pages,2008