The exit times for the diffusion risk model with constant interest

Jingmin He[1, a], Wei Zhang[2,b]

[1]School of Science, Tianjin University of Technology, Tianjin 300384, P.R. China

[2]School of Mathematics and statistics, Central South University, Changsha, Hunan, 410083, China

[a]nkjmhe_2002@163.com, [b]zhangyunfei78@csu.edu.cn

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Abstract. This paper investigates the diffusion risk model with constant interest. The Laplace-Stieltjes transforms (LST) of some exit times of the risk process are obtained.

1 Introduction

The diffusion risk model with constant interest is described by

$$U(t)=u+\int_0^t C(U(s))ds+\sigma B(t), \qquad (1)$$

where u denotes the initial capital, $C(x)=c+rx, c>0$ represents the premiums income pre unit time and $r>0$ is the constant force of interest. $\{B(t),t\geq 0\}$ is a standard Brownian motion and $\sigma>0$ is the diffusion coefficient. The model (1) is a diffusion risk model with interest which represents that the company can earn investment income at a constant force of interest r when the surplus is positive. When the surplus turns negative, the company is allowed to borrow money at the same force of interest r.

For any interval $[b,a]$, where $b<u<a$, define the first hitting time of the upper barrier a for the risk process $\{U(t),t\geq 0\}$ to be

$$T_a=\begin{cases} \inf\{t\geq 0, U(t)=a\}, \\ \infty, \quad if\ U(t)\neq a\ \ for\ all\ t\geq 0. \end{cases}$$

Correspondingly, define the first hitting time of the lower barrier b for the risk process $\{U(t),t\geq 0\}$ to be

$$T_b=\begin{cases} \inf\{t\geq 0, U(t)=b\}, \\ \infty, \quad if\ U(t)\neq b\ \ for\ all\ t\geq 0. \end{cases}$$

Then $T_{a,b}=T_a\wedge T_b$ is the first exit time of the process $\{U(t),t\geq 0\}$ from the interval (b,a). This paper investigates the Laplace-Stieltjes transforms (LST) of the exit times. The similar subject of this paper is considered by some authors. Alili, Patie and Pedersen[1] mainly considered the first hitting time of an Ornstein-Uhlenbeck process. Chiu and Yin[2-4] investigated some passage times of the reserve-dependent risk process and the spectrally negative Lévy process. dos Reis[5] and Gerber[6] mainly studied some stopping times of the classical risk process. Jacobsen and Jensen[7] considered the exit times for a class of piecewise exponential Markov processes with two-sided jumps. Kella and Stadje[8] and Perry and Stadje[9] mainly some exit times of the processes with compound Poisson process.

The remainder of the paper is organized as follows. In section 2 we give some preliminaries of the diffusion process with constant interest. In section 3 we obtain the LST of some exit times.

2 Preliminaries

The model (1) is a time-homogeneous Markov process (see Klebaner[10]) taking values in $\mathbb{R}$ with generator A that satisfies

$$Af(x)=\frac{\sigma^2}{2}f''(x)+(c+rx)f'(x)$$

where f belongs to the domain $D(A)$ of the generator A of $\{U(t),t\ge 0\}$. Furthermore $(U(t),t)$ is also Markovian with generator A' that satisfies

$$A'h(x,t)=Ah(x,t)+\frac{\partial}{\partial t}h(x,t).$$

If $h(x,\cdot)$ has a continuous first derivative for each x and for each t, $h(\cdot,t)$ is in the domain of A, then $h(x,t)\in D(A')$. Denote by $F_t=\sigma\{U(s),0<s\le t\}$ the natural filtration. For later use, we give the following Lemma.

Lemma 2.1 If $h(x,t)$ is a twice continuously differentiable in x and once in t function with bounded first derivative in x, then $h(x,t)\in D(A')$ and furthermore

$$M_h(t)=h(U(t),t)-\int_0^t A'h(U(s),s)ds \qquad (2)$$

is a martingale.

In order to obtain the LST of the first exit time $T_{a,b}$, for any $\alpha>0$, we will try to find a solution to the equation

$$Af(x)=\alpha f(x),$$

that is

$$\frac{\sigma^2}{2}f''(x)+(c+rx)f'(x)=\alpha f(x). \qquad (3)$$

(3) is a second order linear differential equation, which has two positive independent solutions f_1, f_2 such that f_1 is strictly decreasing and f_2 is strictly increasing. Then every solution is a linear combination of the form

$$C_1f_1(x)+C_2f_2(x),$$

where C_1,C_2 are arbitrary constants. From Cai et al.[11], we know that

$$f_1(x)=\exp\{-\frac{(c+rx)^2}{r\sigma^2}\}U(\frac{1}{2}+\frac{\alpha}{2r},\frac{1}{2},\frac{(c+rx)^2}{r\sigma^2}),$$

and

$$f_2(x)=(c+rx)\exp\{-\frac{(c+rx)^2}{r\sigma^2}\}M(1+\frac{\alpha}{2r},\frac{3}{2},\frac{(c+rx)^2}{r\sigma^2}),$$

where M and U are called the confluent hypergeometric functions of the first and second kind respectively. It is easy to verify that $f_1(x)\to 0$ as $x\to+\infty$. More details on confluent hypergeometric functions can be found in Abramowitz and Stegun[12].

3 The LST of some exit times

Theorem 3.1 Given that the initial state $-\frac{c}{r}<a<u$, the LST of the time to hit a is given by

$$E_u[e^{-\alpha T_a}]=\frac{f_1(u)}{f_1(a)}. \qquad (4)$$

Proof. Assume that $h(x,t)$ takes the form $h(x,t)=e^{-\alpha t}f_1(x)$, it follows from Lemma 2.1 that $h(x,t)=e^{-\alpha t}f_1(x)$ is in the domain of A' and

$$A'h(x,t)=Ah(x,t)+\frac{\partial}{\partial t}h(x,t)=0.$$

By Dynkin's formula, we conclude that

$$e^{-\alpha t} f_1(U(t)) - f_1(U(0)) = h(U(t),t) - h(U(0),0) - \int_0^t A'h(U(s),s)ds$$

is a zero-mean martingale. Thus, for stopping time T_a and initial condition u, we have that

$$E_u[e^{-\alpha(t\wedge T_a)} f_1(U(t\wedge T_a))] = f_1(u). \qquad (5)$$

Because $f_1(x)$ is bounded on the range of possible values of $\{U(t\wedge T_a), t\geq 0\}$, letting $t\to +\infty$ in (5), dominated convergence theorem yields

$$E_u[e^{-\alpha T_a} f_1(U(T_a)] = f_1(u),$$

so that

$$E_u[e^{-\alpha T_a}] = \frac{f_1(u)}{f_1(a)}.$$

This completes the proof.

Theorem 3.2 For $b < u < a$, the LST of the first exit time from the upper barrier a is given by

$$E_u[e^{-\alpha T_a} 1(T_a < T_b)] = \frac{f_3(u)}{f_3(a)}, \qquad (6)$$

Where $f_3(x) = C_1 f_1(x) + C_2 f_2(x)$ and C_1, C_2 satisfy $f_3(b) = C_1 f_1(b) + C_2 f_2(b) = 0$.

Proof. It follows from Lemma 2.1 that $h(x,t) = f_3(x)e^{-\alpha t}$ is in the domain of A' and

$$A'h(x,t) = Ah(x,t) + \frac{\partial}{\partial t} h(x,t) = 0.$$

By Dynkin's formula, we conclude that

$$f_3(U(t))e^{-\alpha t} - f_3(U(0)) = h(U(t),t) - h(U(0),0) - \int_0^t A'h(U(s),s)ds$$

is a zero-mean martingale. Thus, for stopping time $T_{a,b}$ and initial condition u, we have that

$$E_u[f_3(U(t\wedge T_{a,b}))e^{-\alpha(t\wedge T_{a,b})}] = f_3(u). \qquad (7)$$

Because $f_3(x)$ is bounded on the range of possible values of $\{U(t\wedge T_{a,b}), t\geq 0\}$, letting $t\to+\infty$ in (7), dominated convergence theorem yields

$$E_u[f_3(U(T_{a,b}))e^{-\alpha T_{a,b}}] = f_3(u),$$

hence

$$E_u[f_3(U(T_a))e^{-\alpha T_a} 1(T_a < T_b)] + E_u[f_3(U(T_b))e^{-\alpha T_b} 1(T_b < T_a)] = f_3(u),$$

thus

$$E_u[f_3(a)e^{-\alpha T_a} 1(T_a < T_b)] = f_3(u),$$

so that

$$E_u[e^{-\alpha T_a} 1(T_a < T_b)] = \frac{f_3(u)}{f_3(a)}.$$

This completes the proof.

Using the same argument, we can obtain the following Theorem.

Theorem 3.3 For $b < u < a$, the LST of the first exit time from the lower barrier b is given by

$$E_u[e^{-\alpha T_b} 1(T_b < T_a)] = \frac{f_4(u)}{f_4(b)}, \qquad (8)$$

Where $f_4(x) = C_1 f_1(x) + C_2 f_2(x)$ and C_1, C_2 satisfy $f_4(a) = C_1 f_1(a) + C_2 f_2(a) = 0$.

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