Split general strong nonlinear quasi-variational inequality problem

Dongxue Han[1,a] , Yali Zhao[2,b*]

[1]College of Mathematics and Physics , Bohai University, Jinzhon , Liaoning 121013,China

[2]College of Mathematics and Physics , Bohai University, Jinzhon , Liaoning 121013, China

[a]email: dongxuehan2014@163.com, [b]email: yalizhao2000@163.com

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Abstract. In this paper, we introduce a split general strong nonlinear quasi-variational inequality problem which is a natural extension of a split general quasi-variational inequality problem, split variational inequality problem, quasi-variational and variational inequality problems in Hilbert spaces. Using the projection method, we propose an iterative algorithm for the split general strongly nonlinear quasi-variational inequality problem and discuss the convergence criteria of the iterative algorithm.The results presented here generalized, unify and improve many previously known results for quasi-variational and variational inequality problems.

Introduction

Variational inequalities are a very powerful tool of the current mathematical technology and have become a rich source of inspiration for scientist and engineers.These have been extended and generalized to study a wide class of problems arising in mechanics, optimization and control problem, operations research and engineering sciences, etc. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the other hand, it enables us to develop highly efficient and powerful numerical methods to solve, for example, obstacle, unilateral, free and moving boundary value problems. In the last five decades, considerable interest has been shown in developing various classes of variational inequality problems, both for their own sake and for their applications.

An important generalization of the variational inequality problem is the quasi-variational inequality problem introduced and studied by Bensoussar, Goursat and Lions [1] in connection with impulse control problem. Recently, Kazmi [2] introduced and studied the following split general quasi-variational inequality problem (in short, SpGQVIP): For each $i \in \{1,2\}$, let $C_i : H_i \to 2^{H_i}$ be a nonempty, closed and convex set-valued mapping, $f_i : H_i \to H_i$ and $g_i : H_i \to H_i$ be nonlinear mapping and let $\mathrm{A} : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Then the SpGQVIP is to find $x_1^* \in H_1$ such that $g_1\left(x_1^*\right) \in C_1\left(x_1^*\right)$ and

$$\langle f_1(x_1^*), x_1 - g_1(x_1^*)\rangle \geq 0 \qquad \text{for all} \quad x_1 \in C_1(x_1^*), \tag{1}$$

and such that $x_2^* = Ax_1^* \in H_2, g_2(x_2^*) \in \mathrm{C}_2(x_2^*)$ solves

$$\langle f_2(x_2^*), x_2 - g_2(x_2^*)\rangle \geq 0 \qquad \text{for all} \quad x_2 \in C_2(x_2^*), \tag{2}$$

SpGQVIP (1)-(2) amounts to saying: find a solution of general quasi-variational inequality GQVI (1) whose image under a given bounded linear operator is a solution of GQVIP (2).

If $g_i = I_i$, where I_i is an identity mapping on H_i, $C_i(x) = \mathrm{C}_i$ for all $x_i \in H_i$, then SpGQVIP (1)-(2) is reduced to the following SpVIP:

Find $x_1^* \in C_1$ such that

$$\langle f_1(x_1^*), x_1 - x_1^*\rangle \geq 0 \quad \text{for all} \quad x_1 \in C_1, \tag{3}$$

and such that $x_2^* = Ax_1^* \in C_2$ solves

$$\langle f_2(x_2^*), x_2 - x_2^* \rangle \geq 0 \quad \text{for all} \quad x_2 \in C_2. \tag{4}$$

SpVIP (3)-(4) has been introduced and studied by Censor, Gibali and Reich [3]. It is worth mentioning that the SpVIP (3)-(4) is quite general and permit split minimization between two spaces so that the imagine of a minimizer of a given function,under a bounded linear operator, is a minimizer of another function and it includes as a special case the split zero problem and the split feasibility problem which have already been studied and used in practice as a model in the intensity-modulated radiation therapy planning, see [4, 5, 6] and the references therein.

In this paper, we introduced the following split general strongly nonlinear quasi-variational inequality problem: For each $i \in \{1,2\}$, let $C_i : H_i \to 2^{H_i}$ be a nonempty,closed and convex set-valued mapping, let $f_i : H_i \to H_i$, $h_i : H_i \to H_i$ and $g_i : H_i \to H_i$ be three nonlinear mapping and let $\mathrm{A} : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Then we consider the problem: Find $x_1^* \in H_1$ such that $g_1\left(x_1^*\right) \in C_1\left(x_1^*\right)$ and

$$\langle f_1(x_1^*) - h_1(x_1^*), x_1 - g_1(x_1^*) \rangle \geq 0 \qquad \text{for all} \quad x_1 \in C_1(x_1^*), \tag{5}$$

and such that $x_2^* = Ax_1^* \in H_2, g_2(x_2^*) \in \mathrm{C}_2(x_2^*)$ solves

$$\langle f_2(x_2^*) - h_2(x_2^*), x_2 - g_2(x_2^*) \rangle \geq 0 \quad \text{for all} \quad x_2 \in C_2(x_2^*). \tag{6}$$

We call problem (5)-(6) the split general strongly nonlinear quasi-variational inequality problem (in short, SpGSNQVIP).

Remark 1. If $h_i = 0$, then SpGSNQVIP (5)-(6) is reduced to SpGQVIP (3)-(4).So the SpGSNQV-IP (5)-(6) is the generalization of SpGQVIP(3)-(4).

Remark 2. Noting that general strongly nonlinear variational inequality problem $\langle f_1(x_1^*) - h_1(x_1^*), x_1 - x_1^* \rangle \geq 0, \forall x_1 \in C_1,$ is a important class of variational inequalities, which is the optimal condition of the following minimization problem:

$$\min_{x \in C}\left(\frac{1}{2}\langle f_1(x), x \rangle - T_1(x)\right),$$

where $T_1(x) = h_1(x)$. we denote the solution set of SpGSNQVIP (5)-(6) and the solution set of SpGQVIP (3)-(4) by Γ_1 and Γ_2, respectively.

Iterative algorithms and convergence results

For each $i \in \{1,2\}$,a mapping P_{C_i} is said to be the metric projection of H_i on C_i if for every point $x_i \in H_i$,there exists a unique nearest point in C_i denoted by $P_{C_i}(x_i)$ such that

$$\left\| x_i - P_{C_i}(x_i) \right\| \leq \left\| x_i - y_i \right\| \qquad \text{for all} \quad y_i \in C_i.$$

It is well known that P_{C_i} is nonexpansive and satisfies

$$\langle x_i - y_i, P_{C_i}(x_i) - P_{C_i}(y_i) \rangle \geq \left\| P_{C_i}(x_i) - P_{C_i}(y_i) \right\|^2 \qquad \text{for all} \quad x_i, y_i \in H_i.$$

Moreover, $P_{C_i}(x_i)$ is characterized by

$$\langle x_i - P_{C_i}(x_i), y_i - P_{C_i}(x_i) \rangle \leq 0 \quad \text{for all} \quad y_i \in C_i.$$

Further, it is easy to see that the following fact: x_1^* satisfied QVIP $\Leftrightarrow$ find $x_1^* \in C_1(x_1^*)$ such that

$$\langle f_1(x_1^*), x_1 - x_1^* \rangle \geq 0 (\forall x_1 \in C_1(x_1^*)) \Leftrightarrow x_1^* = P_{C_1(x_1^*)}(x_1^* - \rho_1 f_1(x_1^*)), \rho_1 > 0.$$

Hence SpGSNQVIP (5)-(6) can be reformulated as follows: Find $x_1^* \in H_1$ with $x_2^* = Ax_1^*$ such that $g_i\left(x_i^*\right) \in C_i\left(x_i^*\right)$ and

$$g_i\left(x_i^*\right) = P_{C_i\left(x_i^*\right)}\left[g_i\left(x_i^*\right) - \rho_i (f_i(x_i^*) - h_i(x_i^*)) \right],$$

for $\rho_i > 0$.

Based on the above arguments, we propose the following iterative algorithm for approximating a solution to SpGSNQVIP (5)-6).

Let $\{\alpha^n\} \subset (0,1)$ be a sequence such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, and let ρ_1, ρ_2, γ be the parameters with positive values.

Algorithm 1. Given $x_1^0 \in H_1$, compute the iterative sequence $\{x_1^n\}$ by the iterative schemes:

$$g_1(y^n) = P_{C_1(x_1^n)}\left[g_1(x_1^n) - \rho_1(f_1(x_1^n) - h_1(x_1^n))\right], \tag{7}$$

$$g_2(z^n) = P_{C_2(Ay^n)}\left[g_2(Ay^n) - \rho_2(f_2(Ay^n) - h_2(Ay^n))\right], \tag{8}$$

$$x_1^{n+1} = (1-\alpha_n)x_1^n + \alpha_n\left[y^n + \gamma A^*(z^n - \mathrm{A}\, y^n)\right] \tag{9}$$

for all $n = 0,1,2,\cdots, \rho_1, \rho_2, \gamma > 0.$

If $H_i = 0$,then Algorithm 1 is reduced to the following iterative algorithm for SpGQVIP (3)-(4):

Algorithm 2 . Given $x_1^0 \in H_1$, compute the iterative sequence $\{x_1^n\}$ by the iterative schemes:

$$g_1(y^n) = P_{C_1(x_1^n)}\left[g_1(x_1^n) - \rho_1 f_1(x_1^n)\right], \tag{10}$$

$$g_2(z^n) = P_{C_2(Ay^n)}\left[g_2(Ay^n) - \rho_2 f_2(Ay^n)\right], \tag{11}$$

$$x_1^{n+1} = (1-\alpha_n)x_1^n + \alpha_n\left[y^n + \gamma A^*(z^n - \mathrm{A}\, y^n)\right] \tag{12}$$

for all $n = 0,1,2,\cdots, \rho_1, \rho_2, \gamma > 0.$

If $g_i = I_i, C_{i(x_i)} = C_i (\forall x_i \in H_i)$,where C_i is a nonempty closed convex subset of H_i, then Algorithm 2 is reduced to the following iterative algorithm for SpVIP (1)-(2):

Algorithm 3. Given $x_1^0 \in H_1$, compute the iterative sequence $\{x_1^n\}$ by the iterative schemes:

$$y^n = P_{C_1}\left[x_1^n - \rho_1 f_1(x_1^n)\right], \tag{13}$$

$$z^n = P_{C_2}\left[Ay^n - \rho_2 f_2(Ay^n)\right], \tag{14}$$

$$x_1^{n+1} = (1-\alpha_n)x_1^n + \alpha_n\left[y^n + \gamma A^*(z^n - \mathrm{A}\, y^n)\right] \tag{15}$$

for all $n = 0,1,2,\cdots, \rho_1, \rho_2, \gamma > 0.$

Remark 3. Algorithm2 and Algorithm3 are proposed by Kazmi in [2] and [7], respectively. Noting that Algorithm1 concludes them as special cases.

In order to obtain our main results, we need the following assumption, definition and lemmas.

Assumption 1. For all $x_i, y_i, z_i \in H_i$,the operator $P_{C_i(x_i)}$ satisfies the condition:

$$\left\|P_{C_i(x_i)}(z_i) - P_{C_i(y_i)}(z_i)\right\| \le v_i \left\|x_i - y_i\right\|$$

for some constant $v_i > 0$.

Definition 1. A nonlinear mapping $f_1 : H_1 \to H_1$ is said to be

(i) $\alpha_1 -$strongly monotone with respect to $g_1 : H_1 \to H_1$ if there exists a constant $\alpha_1 > 0$ such that $\langle f_1(x) - f_1(y), \mathrm{g}_1(x) - g_1(y)\rangle \ge \alpha_1 \left\|x - y\right\|^2, \forall x, y \in H_1;$

(ii) $\beta_1 -$ Lipschitz continuous if there exists a constant $\beta_1 > 0$ such that $\left\| f_1(x) - f_1(y)\right\| \le \beta_1 \left\|x - y\right\|, \forall x, y \in H.$

Remark 4. If $g_1 = I_1$,where I_1 is an identity mapping on H_1, then definition 1(i) is reduced to the definition of $\alpha_1 -$strongly monotone of f .

Lemma 1.. Let H be a real Hilbert space. Then the following inequalities hold:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \forall x, y \in H;\ \langle x, y\rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x-y\|^2), \forall x, y \in H.$$

Lemma 2[8]. Assume that $\{a_n\}$ is a sequence of nonnegative numbers such that $a_{n+1} \le (1-\gamma_n)a_n + \delta_n$, where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that:

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$; (ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. Then $\lim_{n\to\infty} \alpha_n = 0$.

Now we study the convergence of Algorithm1 for SpGSNQVIP (5)-(6).

Theorem 1. For each $i \in \{1,2\}$, let $C_i : H_i \to 2^{H_i}$ $\frac{1}{2}$ be a nonempty, closed and convex set-valued mapping, let $g_i : H_i \to H_i$ be $\delta_i -$ Lipschitz continuous such that $(g_i - I_i)$ is $\sigma_i -$ strongly monotone, where I_i is the identity mapping on H_i. Let $f_i : H_i \to H_i$ be $\alpha_i -$ strongly monotone with respect to g_i and $\beta_i -$ Lipschitz continuous. Let $h_i : H_i \to H_i$ be $\xi_i -$ Lipschitz continuous and let $A : H_1 \to H_2$ be a bounded linear operator and A^* be its adjoint mapping. Suppose $x_1^* \in H_1$ is a solution to SpGSNQVIP (5)-(6) and Assumption 1 holds.Then the sequence $\{x_1^n\}$ generated by Algorithm 1 convergences strongly to x_1^* provided that the constants ρ_i and γ satisfy the following conditions:

$$\left|\rho_1 - \frac{\alpha_1 - k_1\xi_1}{\beta_1^2 - \xi_1^2}\right| \le \frac{\sqrt{(\alpha_1 - k_1\xi_1)^2 - (\delta_1^2 - k_1^2)(\beta_1^2 - \xi_1^2)}}{\beta_1^2 - \xi_1^2}, k_1 = \frac{\sqrt{1+2\sigma_1}}{1+2\theta_2} - v_1, \quad ,$$

$$|\alpha_1 - k_1\xi_1| > \sqrt{(\delta_1^2 - k_1^2)(\beta_1^2 - \xi_1^2)}, \delta_1 > |k_1|, \beta_1 > \xi_1,$$

$$0 < \theta_2 = \frac{1}{\sqrt{1+2\sigma_2}}\left\{v_2 + \sqrt{\sigma_2 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2} + \rho_2\xi_2\right\}, \rho_2 > 0, \gamma \in (0, \frac{2}{\|A\|^2}).$$

Proof. Since $x_1^* \in H_1$ is a solution of SpGSNQVIP (5)-(6), $x_1^* \in H_1$ is such that $g_i(x_i^*) \in C_i(x_i^*)$ and

$$g_1(x_1^*) = P_{C_1(x_1^*)}\left[g_1(x_1^*) - \rho_1(f_1(x_1^*) - h_1(x_1^*))\right], \tag{16}$$

$$g_2\left(Ax_1^*\right) = P_{C_2(Ax_1^*)}\left[g_2\left(Ax_1^*\right) - \rho_2(f_2(Ax_1^*) - h_2(Ax_1^*))\right], \tag{17}$$

for $\rho_i > 0$. It follows from Algorithm 1(7), Assumption 1 and (16) that

$$\|g_1(y^n) - g_1(x_1^*)\| = \left\|P_{C_1(x_1^n)}\left[g_1(x_1^n) - \rho_1(f_1(x_1^n) - h_1(x_1^n))\right] - P_{C_1(x_1^*)}\left[g_1(x_1^*) - \rho_1(f_1(x_1^*) - h_1(x_1^*))\right]\right\|$$

$$\le \left\|P_{C_1(x_1^n)}\left[g_1(x_1^n) - \rho_1(f_1(x_1^n) - h_1(x_1^n))\right] - P_{C_1(x_1^n)}\left[g_1(x_1^*) - \rho_1(f_1(x_1^*) - h_1(x_1^*))\right]\right\|$$

$$+ \left\|P_{C_1(x_1^n)}\left[g_1(x_1^*) - \rho_1(f_1(x_1^*) - h_1(x_1^*))\right] - P_{C_1(x_1^*)}\left[g_1(x_1^*) - \rho_1(f_1(x_1^*) - h_1(x_1^*))\right]\right\|$$

$$\le \|g_1(x_1^n) - g_1(x_1^*) - \rho_1(f_1(x_1^n) - f_1(x_1^*))\| + \rho_1\|h_1(x_1^n) - h_1(x_1^*)\| + v_1\|x_1^n - x_1^*\| \tag{18}$$

Noting that f_1 is $\alpha_1 -$ strongly monotone with respect to g_1 and β_1-Lipschitz continuous and g_1 is δ_1-Lipschitz continuous, we have

$$\|g_1(x_1^n) - g_1(x_1^*) - \rho_1(f_1(x_1^n) - f_1(x_1^*))\|^2$$

$$= \|g_1(x_1^n) - g_1(x_1^*)\|^2 - 2\rho_1\langle f_1(x_1^n) - f_1(x_1^*), g_1(x_1^n) - g_1(x_1^*)\rangle + \rho_1^2\|f_1(x_1^n) - f_1(x_1^*)\|^2$$

$$\le (\delta_1^2 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2)\|x_1^n - x_1^*\|^2 \tag{19}$$

Combining (18) and (19), we get

$$\left\|g_1(y^n)-g_1(x_1^*)\right\| \le \left\{\sqrt{\delta_1^{\,2}-2\rho_1\alpha_1+\rho_1^{\,2}\beta_1^{\,2}}+\rho_1\xi_1+v_1\right\}\left\|x_1^n-x_1^*\right\| \tag{20}$$

Since $(\mathrm{g}_1-\mathrm{I}_1)$ is σ_1-strongly monotone, by virtue of Lemma 1(1), we have

$$\begin{aligned}\left\|y^n-x_1^*\right\|^2 &\le \left\|g_1(y^n)-g_1(x_1^*)\right\|^2-2\langle(\mathrm{g}_1-\mathrm{I}_1)y^n-(\mathrm{g}_1-\mathrm{I}_1)x_1^*, y^n-x_1^*\rangle\\ &\le \left\|g_1(y^n)-g_1(x_1^*)\right\|^2-2\sigma_1\left\|y^n-x_1^*\right\|^2,\end{aligned}$$

Which implies that

$$\left\|y^n-x_1^*\right\|^2 \le \frac{1}{\sqrt{1+2\sigma_1}}\left\|g_1(y^n)-g_1(x_1^*)\right\|. \tag{21}$$

It follows from (20) and (21), we have

$$\left\|y^n-x_1^*\right\| \le \theta_1\left\|x_1^n-x_1^*\right\|. \tag{22}$$

where $\theta_1=\dfrac{1}{\sqrt{1+2\sigma_1}}\left\{\sqrt{\delta_1^{\,2}-2\rho_1\alpha_1+\rho_1^{\,2}\beta_1^{\,2}}+\rho_1\xi_1+v_1\right\}$.

Similarly, we obtain

$$\left\|g_2(z^n)-g_2(Ax_1^*)\right\| \le \left\{\sqrt{\delta_2^{\,2}-2\rho_2\alpha_2+\rho_2^{\,2}\beta_2^{\,2}}+\rho_2\xi_2+v_2\right\}\left\|Ay^n-Ax_1^*\right\| \tag{23}$$

and

$$\left\|z^n-Ax_1^*\right\| \le \theta_2\left\|Ay^n-Ax_1^*\right\|. \tag{24}$$

Where $\theta_2=\dfrac{1}{\sqrt{1+2\sigma_2}}\left\{\sqrt{\delta_2^{\,2}-2\rho_2\alpha_2+\rho_2^{\,2}\beta_2^{\,2}}+\rho_2\xi_2+v_2\right\}$. Furthermore, in view of Algorithm 1(9), we have

$$\left\|x_1^{n+1}-x_1^*\right\| \le (1-\alpha_n)\left\|x_1^n-x_1^*\right\|+\alpha_n\left[\left\|y^n-x_1^*-\gamma A^*(Ay^n-Ax_1^*)\right\|+\gamma\left\|A^*(z^n-Ax_1^*\right\|\right]. \tag{25}$$

Observe that A^* is a bounded linear operator with $\|A\|=\|A^*\|$ and the given condition on γ, we get

$$\begin{aligned}\left\|y^n-x_1^*-\gamma A^*(Ay^n-Ax_1^*)\right\|^2 &= \left\|y^n-x_1^*\right\|^2-2\gamma\langle y^n-x_1^*, A^*(Ay^n-Ax_1^*)\rangle+\gamma^2\left\|A^*(Ay^n-Ax_1^*\right\|^2\\ &\le \left\|y^n-x_1^*\right\|^2-\gamma(2-\gamma\|A\|^2)\left\|Ay^n-Ax_1^*\right\|^2 \le \left\|y^n-x_1^*\right\|^2.\end{aligned} \tag{26}$$

And using (24), we have

$$\left\|A^*(z_n-Ax_1^*)\right\| \le \|A\|\left\|z_n-Ax_1^*\right\| \le \theta_2\|A\|\left\|Ay^n-Ax_1^*\right\| \le \theta_2\|A\|^2\left\|y^n-x_1^*\right\|. \tag{27}$$

From (25)-(27) that

$$\left\|x_1^{n+1}-x_1^*\right\| \le \left[1-(1-\theta)\alpha^n\right]\left\|x_1^n-x_1^*\right\|.$$

Where $\theta=\theta_1(1+\gamma\|A\|^2\theta_2)$. It follows from the conditions on ρ_1,ρ_2 and γ that $\theta\in(0,1)$. Thus $\left\{(1-\theta)\alpha^n\right\}\subset(0,1)$ and $\sum_{n=1}^{\infty}(1-\theta)\alpha^n=\infty$ for $\sum_{n=1}^{\infty}\alpha^n=\infty$. So it follows from Lemma 2 that $\left\{x_1^n\right\}$ converges strongly to x_1^* as $n\to\infty$. Since A is continuous, it follows from(20), (22), (23) and (24) that $g_1(y^n)\to g_1(x_1^*), y^n\to x_1^*, Ay^n\to Ax_1^*$, $g_2(z^n)\to g_2(Ax_1^*)$, and $z^n\to Ax_1^*$ as $n\to\infty$. This completes the proof.

If $h_i=0$, then Theorem 1 reduced to the following result of the convergence of Algorithm 2 for SpGQVIP (10)-(11).

Corollary 1. For each $i\in\{1,2\}$, let $C_i:H_i\to 2^{H_i}$ be a nonempty,closed and convex set-valued mapping, let $g_i:H_i\to H_i$ be δ_i-Lipschitz continuous such that $(\mathrm{g}_i-\mathrm{I}_i)$ is σ_i-strongly monotone, where I_i is the identity mapping on H_i. Let $f_i:H_i\to H_i$ be α_i-strongly monotone with respect

to g_i and β_i- Lipschitz continuous. Let $A: H_1 \to H_2$ be a bounded linear operator and A^* be its adjoint mapping. Suppose $x_1^* \in H_1$ is a solution to SpGQVIP (1)-(2) and Assumption 1 holds.Then the sequence $\{x_1^n\}$ generated by Algorithm 2 convergences strongly to x_1^* provided that the constants ρ_i and γ satisfy the conditions:

$$\left|\rho_1 - \frac{\alpha_1}{\beta_1^2}\right| \le \frac{\sqrt{\alpha_1^2 - \beta_1^2(\delta_1^2 - k_1^2)}}{\beta_1^2}, \alpha_1 > \beta_1\sqrt{\delta_1^2 - k_1^2}, \quad k_1 = \left[\frac{\sqrt{1+2\sigma_1}}{\sqrt{1+2\theta_2}} - v_1\right], \delta_1 > |k_1|,$$

$$0 < \theta_2 = \frac{1}{\sqrt{1+2\sigma_2}}\left\{v_2 + \sqrt{\sigma_2^2 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2}\right\}, \rho_2 > 0, \gamma \in (0, \frac{2}{\|A\|^2}).$$

If $C_{i(x_i)} = C_i (\forall x_i \in H_i)$, where C_i is a nonempty closed and convex subset of H_i, $g_i = I_i, h_i = 0$, then Theorem 1 reduces to the following convergence result of Algorithm 3 for SpVIP(3)-(4).

Corollary 2. For each $i \in \{1,2\}$, let C_i be a nonempty,closed and convex subset of H_i. Let $f_i : H_i \to H_i$ be α_i- strongly monotone and β_i- Lipschitz continuous. Let $A: H_1 \to H_2$ be a bounded linear operator and A^* be its adjoint operators. Suppose $x_1^* \in H_1$ is a solution to SpVIP (3)-(4). Then the sequence $\{x_1^n\}$ generated by Algorithm 3 converges strongly to x_1^* provided that the constants ρ_i and γ satisfy the conditions:

$$\left|\rho_1 - \frac{\alpha_1}{\beta_1^2}\right| \le \frac{\sqrt{\alpha_1^2 - \beta_1^2(1-k_1^2)}}{\beta_1^2}, \alpha_1 > \beta_1\sqrt{1-k_1^2}, k_1 = \frac{1}{1+2\theta_2}, k_1 < 1, \quad \theta_2 = \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2}.$$

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