

Global minimization with a new filled function approach

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Abstract. This paper presents a new filled function method for global minimization problem with smooth or non-smooth box constraints. The constructed filled function contains only one parameter, which can be adjusted readily during the iterative process. The theories and the computational properties of the filled function are investigated, and a corresponding filled function approach is proposed. At last, a few numerical experiments, including the application of the filled function method to solution of nonlinear equations, are reported.

Introduction

More and more issues arising from science and engineering fields can be modelled as nonlinear global minimization problems, and global minimization now is being regarded as one of the most hot branch in optimization. Lots of extant theories and corresponding algorithms could be invoked to solve these global minimization problems, including both stochastic ones and deterministic ones. For general global minimization problems with no specific structure, the filled function method, one category of efficient deterministic methods, could be applied. The filled function method was proposed firstly by Ge [1] for addressing smooth global optimization problem $(P): \min_{x \in X} f(x)$, where X is a box set, and the idea of this method is that the objective function manages to escape from a given local minimizer x^* through the utilization of an auxiliary function called filled function. The filled function method generally consists of two stages. The aim of the stage 1 is to search for one of local minimizers of (P) via any efficient local minimization algorithm. Then filled function method switches to phase 2, if the stage 1 has finished. Stage 2 constructs and minimizes a filled function, and then get an better starting point (with lower objective function value) for stage 1. These two stages are performed repeatedly, until no improved minimizers be found. The filled function proposed in [1] contains an exponential term, which might lead to the failure of calculation when the values of the exponential term increases rapidly. In order to deal with this kind of problem, the authors of the papers [2,3,4,5,6] reconsidered the filled function. The filled function defined in [2] improved the one in [1], but its requirement that the minimization of filled function should be along a line connecting the current optimizer and a point in some neighborhoods of the next better optimizer which is unknown, is hard to be satisfied, since searching for such a direction will result in numerical complexity during computation. The filled functions constructed in [3,4,5,6] confine themselves to smooth global minimization problems. Although these algorithms have many advantages, they have their own limitations, for many real-life problems are usually modelled as non-smooth global minimization problems rather than smooth ones. In this paper, we draw inspiration from these filled function methods, combine their merits and propose a new filled function which is applicable to both smooth and non-smooth global minimization problems.

This paper is organized as follows: Following this introduction, in Section 2, we will give basic knowledge that is necessary for the subsequent sections, and then construct a filled function and discuss its properties in detail. In Section 3, we establish an corresponding filled function algorithm.

In Section 4, we provide several numerical results, including the application of the filled function method in solving nonlinear equations.

A filled function and its properties

In this paper, we consider the following box-constrained global minimization problem $(P): \min_{x \in X} f(x)$, where X is a box set. For simplicity, we denote the set of local minimizers of problem (P) by $L(P)$, impose two conditions on function $f(x)$, and give the definition a filled function for problem (P) . Let $x^* \in L(P)$.

To begin with, we make the following assumptions:

Assumption 1. The function $f(x)$ is Lipschitz continuous on X with a rank $L > 0$.

Assumption 2. The problem (P) has at least one global minimizer and the number of the different minimal values is finite.

The main tools used by filled function method for non-smooth global minimization is Clark generalized gradient. For more details of the Clark generalized gradient, please refers to [9].

Now, we give the definition of filled function for non-smooth global minimization problem (P) .

Definition. A function $P(x, x^*)$ is said to be a filled function of problem (P) at $x^* \in L(P)$, if it satisfies following conditions:

- (1) x^* is a strictly maximizer of $P(x, x^*)$ on X .
- (2) For any point $x^* \neq x \in X$ satisfying that $f(x) \geq f(x^*)$, $0 \notin \partial P(x, x^*)$.
- (3) If x^* is not a global minimizer of $f(x)$, then there exists $x_0 \in S_2 = \{x \in X : f(x) < f(x^*)\}$

such that x_0 is a local minimizer of $P(x, x^*)$.

$$\text{Define } g_r(t) = \begin{cases} 1 & t > 0 \\ \sin 0.5\pi(1 + \frac{t}{r})^2 & -r < t \leq 0 \\ 0 & t \leq -r \end{cases}$$

$$h_r(t) = \begin{cases} 1 & t > 0 \\ (r-2)(\frac{t}{r})^3 + (r-3)(\frac{t}{r})^2 + 1 & -r < t \leq 0 \\ t+r & t \leq -r \end{cases}$$

Then, it can be easily verified that both the above functions are continuously differentiable with their respective derivatives given as follows.

$$g_r'(t) = \begin{cases} 0 & t > 0 \\ \frac{\pi}{r}(1 + \frac{t}{r}) \cos 0.5\pi(1 + \frac{t}{r})^2 & -r < t \leq 0 \\ 0 & t \leq -r \end{cases}$$

$$h_r'(t) = \begin{cases} 0 & t > 0 \\ 3(1 - \frac{2}{r})(\frac{t}{r})^2 + 2(1 - \frac{3}{r})(\frac{t}{r}) & -r < t \leq 0 \\ 1 & t \leq -r \end{cases}$$

Now, we establish a new filled function as follows:

$$P(x, x^*, r) = e^{-\|x-x^*\|} g_r(f(x) - f(x^*)) + h_r(f(x) - f(x^*)). \quad (1)$$

where $r > 0$ is a parameter.

The following theorems show that $P(x, x^*, r)$ is a filled function.

Theorem 1. Let $x^* \in L(P)$, then x^* is a strict local maximizer of $P(x, x^*, r)$.

Proof. Since $x^* \in L(P)$, there exists a neighborhood $N(x^*, \sigma^*)$ of x^* such that $f(x) \geq f(x^*)$ for all $x \in N(x^*, \sigma^*) \cap X$, where $0 < \sigma^* < 1$ is a constant. It holds that

$$P(x, x^*, r) = e^{-\|x-x^*\|} + 1 < 1 + 1 = P(x^*, x^*, r), \text{ for } x^* \neq x \in N(x^*, \sigma^*) \cap X. \quad (2)$$

Thus, x^* is a strict local maximizer of $P(x, x^*, r)$.

Theorem 2. Let $x^* \in L(P)$, then, for any $x \neq x^*$ and $f(x) \geq f(x^*)$, we have $0 \notin \partial P(x, x^*, r)$.

Proof. By the conditions, it holds that

$$\partial P(x, x^*, r) \subseteq -\frac{(x-x^*)}{\|x-x^*\|} e^{-\|x-x^*\|}. \quad (3)$$

Thus, we have

$$\langle x-x^*, \partial P(x, x^*, r) \rangle \leq -\|x-x^*\| e^{-\|x-x^*\|} < 0. \quad (4)$$

Hence, we have $0 \notin \partial P(x, x^*, r)$.

Theorem 3. Assume that $x^* \in L(P)$, but it is not a global minimizer, then there exists a point $x_0 \in S_2 = \{x \in X : f(x) < f(x^*)\}$ such that x_0 is a local minimizer of $P(x, x^*, r)$, and $P(x_0, x^*, r) < P(x^*, x^*, r)$, $P(x_0, x^*, r) < P(x, x^*, r)$ for any $x \in \partial X$, where ∂X is the boundary of X .

Proof. Denote $d = \min_{x_1, x_2 \in L(P), f(x_1) \neq f(x_2)} |f(x_1) - f(x_2)|$, $0 < r < d$.

Since x^* is not a global minimizer of (P), there exists another minimizer x_0 , such that $f(x^*) > f(x_0)$. Thus, we have $f(x_0) - f(x^*) \leq -d < -r$.

By the continuity of $f(x)$, there exists one neighborhood $N(x_0, \sigma)$ of x_0 , such that $f(x) - f(x^*) < -r$, for any $x \in N(x_0, \sigma)$. Therefore $g_r(f(x) - f(x^*)) = 0$, $\forall x \in N(x_0, \sigma)$.

Since x_0 is a minimizer of (P), there exists one neighborhood $N(x_0, \sigma_1)$, such that $f(x) \geq f(x_0)$, for any $x \in N(x_0, \sigma_1)$. Let $\sigma_2 < \min(\sigma_1, \sigma)$, then, for any $x \in N(x_0, \sigma_2)$, we have

$P(x, x^*, r) = f(x) - f(x^*) + r \geq f(x_0) - f(x^*) + r = P(x_0, x^*, r)$. Thus, x_0 is a minimizer of $P(x, x^*, r)$. Moreover, it holds

$$P(x_0, x^*, r) = f(x_0) - f(x^*) + r < 0 < 1 + 1 = P(x^*, x^*, r),$$

$$P(x_0, x^*, r) = f(x_0) - f(x^*) + r < 0 < e^{-\|x-x^*\|} + 1 = P(x, x^*, r), \forall x \in \partial X.$$

Theorem 4. Assume that $x^* \in L(P)$, if $\|x_2 - x^*\| > \|x_1 - x^*\|$, $f(x_2) \geq f(x^*)$, $f(x_1) \geq f(x^*)$, then $P(x_2, x^*, r) < P(x_1, x^*, r)$.

Proof. By the conditions, we have

$$P(x_1, x^*, r) = e^{-\|x_1-x^*\|} + 1 > P(x_2, x^*, r) = e^{-\|x_2-x^*\|} + 1. \quad (5)$$

This completes the proof of the theorem.

Filled function algorithm

In the above section, we discussed some properties of the filled function. Now, we give a filled function algorithm below.

Filled function algorithm

Initialization step:

Let r_l the lower bound of parameter r , x_1 the initial point and e_1, e_2, \dots, e_{2n} the positive and negative coordinate directions. Set $k = 1$, and go to the main step.

Main step

1. Starting from x_1 , minimize (P) by any non-smooth local minimization procedure to find a local minimizer x_1^* and go to 2.

2. Set $r = 1$.

3. Construct a filled function $P(x, x_1^*, r)$ and go to 3.

4. If $k > 2n$, then go to 7; otherwise, set $x = x_1^* + 0.1e_k$, and take x as an initial point to find a local minimizer x_k of the following problem: $\min_{y \in X} P(y, x_1^*, r)$.

5. If $x_k \notin X$, then set $k = k + 1$, and go to 4; otherwise, go to 6.

6. If $f(x_k) < f(x_1^*)$, then, (a) set $x = x_k$, $k = 1$. (b) Use x as a new initial point and minimize (P) to find its another local minimizer x_2^* with $f(x_2^*) < f(x_1^*)$. (c) Set $x_1^* = x_2^*$ and go to 2; Else if $f(x_k) \geq f(x_1^*)$, then go to 7.

7. Decrease r by setting $r = 0.1r$. If $r > r_l$, then set $k = 1$, and go to 3; otherwise, take x_1^* as a global minimizer, and the algorithm stops.

Remarks:

(1) The presented filled function algorithm is also suitable for solving smooth box-constrained global minimization problem.

(2) The filled function method mainly consists of two stages, that is, stage 1: local minimization and stage 2: filling stage. In stage 1, a local minimizer x^* is found by any non-smooth local minimization algorithm, such as Hybrid Hooke and Jeeves-Direct Method for Non-smooth Optimization [8], Mesh Adaptive Direct Search Algorithms for Constrained Optimization [7], Bundle methods, Powell's method, etc. In particular, the Hybrid Hooke and Jeeves-Direct Method is more preferable, since it guarantees that a local minimizer can be found at last. In stage 2, the constructed filled function $P(x, x^*, r)$ is minimized. During the process of minimization, if a point x_k with $f(x_k) < f(x^*)$ is found, then the stage 2 stops and the algorithm switches to the stage 1 to find a better optimizer with lower function value for $f(x)$. The above two stages repeat until one global minimizer is found.

Numerical experiment

The proposed filled function method has lots of applications. In this section, we perform a few numerical tests, including the application of the filled function method to nonlinear equations.

Problem 1:

$$\min f(x) = \left| \frac{x-1}{4} \right| + \left| \sin\left(\pi\left(1 + \frac{x-1}{4}\right)\right) \right| + 7, |x| \leq 10.$$

The algorithm successfully found a global solution: $x^* = 1$ with $f(x^*) = 7$. Table 1 records the numerical results of Problem 1.

Problem 2:

$$\min f(x) = \max\{5x_1 + x_2, -5x_1 + x_2, x_1^2 + x_2^2 + 4x_2\}, -4 \leq x_1, x_2 \leq 4.$$

The algorithm successfully found a global solution: $x^* = (0, -3)$ with $f(x^*) = -3$. Table 2 records the numerical results of Problem 2.

The idea of application of filled function method to solution of nonlinear equations is given as follows:

Consider the following nonlinear equations (NE): $G(x) = 0$, $x \in X$, where the mapping $G(x) = (f_1(x), f_2(x), \dots, f_m(x))^T : R^n \rightarrow R^m$ is continuous, and $X \subset R^n$ is a box set.

Let $f(x) = \sum_{k=1}^m |f_k(x)|$, then the solution of problem (NE) may be obtained through solving the following reformulated global optimization problem (P): $\min_{x \in X} f(x)$. In particular, suppose that the problem (NE) has at least one root, then each global minimizer of the problem (P) with zero function value corresponds to one root of the (NE).

Problem 3:

$$10^4 x_1 x_2 = 1, e^{-x_1} + e^{-x_2} = 1.001,$$

$$s.t. 5.49 \times 10^{-6} \leq x_1 \leq 4.553, 2.196 \times 10^{-3} \leq x_2 \leq 18.21.$$

The algorithm successfully found its solution $x^* = (1.450 \times 10^{-5}, 6.8933335)$. Table 3 records its numerical results.

Problem 4:

$$2x_1 + x_2 + x_3 + x_4 + x_5 = 6, x_1 + 2x_2 + x_3 + x_4 + x_5 = 6, x_1 + x_2 + 2x_3 + x_4 + x_5 = 6,$$

$$x_1 + x_2 + x_3 + 2x_4 + x_5 = 6, x_1 x_2 x_3 x_4 x_5 = 1$$

$$s.t. |x_i| \leq 2, i = 1, 2, \dots, 5.$$

The known solutions of the Problem 4 are (1,1,1,1,1) and (0.916,0.916,0.916,0.916,0.916). Table 4 records its numerical results.

Table 1: Computational results for Problem 1

k	x_k^0	$f(x_k)$	x_k^*	$f(x_k^*)$
1	6.0000	8.9571	5.0000	8.0001
2	0.9678	7.0333	0.9998	7.0001

Table 2: Computational results for Problem 2

k	x_k^0	$f(x_k)$	x_k^*	$f(x_k^*)$
1	(1,1)	6.0000	(0.0000,0.0000)	0.0000
2	(-0.0002,-0.9725)	-0.9715	(-0.0002,-0.9725)	-0.9715
3	(-0.0003,-2.5644)	-2.5487	(0.0000,-3.0000)	-3.0000

Table 3: Computational results for Problem 3

k	x_k^0	x_k^*	$f(x_k^*)$	$G(x_k^*)$
1	$\begin{pmatrix} 3.0000 \\ 3.0000 \end{pmatrix}$	$\begin{pmatrix} 0.00001457 \\ 6.87403875 \end{pmatrix}$	7.11982×10^{-6}	$\begin{pmatrix} -4.6287 \times 10^{-6} \\ -2.4911 \times 10^{-6} \end{pmatrix}$
2	$\begin{pmatrix} 1.4523 \times 10^{-5} \\ 6.89330451 \end{pmatrix}$	$\begin{pmatrix} 0.000014509 \\ 6.89330448 \end{pmatrix}$	0.00000003	$\begin{pmatrix} 0.00000000 \\ 0.00000003 \end{pmatrix}$

Table 4: Computational results for Problem 4

k	x_k^0	x_k^*	$f(x_k^*)$	$G(x_k^*)$
1	$\begin{pmatrix} 1.5000 \\ 1.5000 \\ 1.5000 \\ 1.5000 \\ 1.5000 \end{pmatrix}$	$\begin{pmatrix} 1.01133928 \\ 1.01132927 \\ 1.01134016 \\ 1.01134121 \\ 0.94730728 \end{pmatrix}$	4.0921×10^{-3}	$\begin{pmatrix} 4.00334 \times 10^{-3} \\ 4.00332 \times 10^{-3} \\ 4.00405 \times 10^{-3} \\ 4.00431 \times 10^{-3} \\ -8.98971 \times 10^{-5} \end{pmatrix}$
2	$\begin{pmatrix} 0.92448461 \\ 0.92009047 \\ 0.91667846 \\ 0.91421239 \\ 1.40388963 \end{pmatrix}$	$\begin{pmatrix} 0.92448449 \\ 0.92009082 \\ 0.91667851 \\ 0.91421226 \\ 1.40388973 \end{pmatrix}$	1.7952×10^{-3}	$\begin{pmatrix} 0.00384037 \\ -0.00055348 \\ -0.00396568 \\ -0.00643192 \\ 0.00075419 \end{pmatrix}$

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