Existence of Positive Periodic Solution of a Ratio-Dependent Predator- Prey System with Time Delays

Xinyi Chen

China Minorities Information Technology Institute, Northwest Minzu University, Lanzhou 730030, China

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Abstract. A periodic ratio-dependent predator-prey system with time delay is investigated. By using a continuation theorem based on coincidence degree theory, the sufficient conditions of the existence of periodic solution of the system are obtained, which generalizes the known result.

Introduction

Ratio-dependent predator-prey models have received much attention recently as more suitable ones for predator-prey interactions where predation involves searching process. A typical ratio-dependent predator-prey model can be express in the form

$$\begin{cases} y_1'(t) = y_1(t)(a - by_1(t) - \dfrac{cy_2(t)}{my_2(t) + y_1(t)}], \\ y_2'(t) = y_2(t)[-d + \dfrac{fy_1(t)}{my_2(t) + y_1(t)}], \end{cases} \tag{1}$$

where y_1 and y_2 stand for prey and predator density, respectively. a, b, c, d, f and m are positive constants. For the ecological sense of system (1), we refer to [1] and reference therein. System (1) was systematically studied by Kuang and Beretta [1] and Arditi and coworkers[2-7]. They discussed global stability of the boundary equilibria, positive equilibrium, and permanence of the system. Therefore, paper [8] is interesting and important to study the following periodic ratio-dependent system with time delays

$$\begin{cases} y_1'(t) = y_1(t)[a(t) - b(t)y_1(t-\tau_1) - \dfrac{c(t)y_2(t)}{m(t)y_2(t) + y_1(t)}], \\ y_2'(t) = y_2(t)[-d(t) + \dfrac{f(t)y_1(t-\tau_2)}{m(t)y_2(t-\tau_2) + y_1(t-\tau_2)}]. \end{cases} \tag{2}$$

with initial conditions

$$y_i(s) = \varphi_i(s), s \in [-\tau, 0], \varphi_i(s) > 0, \varphi_i \in C([-\tau, 0], R_+), i = 1,2. \tag{3}$$

where $a(t), b(t), c(t), m(t), d(t)$ and $f(t)$ are strictly positive continuous ω-periodic functions. τ_1 and τ_2 are nonnegative constants, $\tau = \max\{\tau_1, \tau_2\}$. They obtained the sufficient conditions of the positive periodic solution of the system as follows

Theorem1.1 Assume the following conditions are satisfied

(H$_1$) $\overline{f(t) - d(t)} > 0$,

(H$_2$) $\overline{ma - c} > 0$.

Then system (2) has at least one positive ω-periodic solution.

We will be concerned with a more general system as follows

$$\begin{cases} y_1'(t) = y_1(t)[a(t) - b(t)y_1(t-\tau_1) - \dfrac{c(t)y_2^p(t)}{m(t)y_2^p(t) + y_1^p(t)}], \\ y_2'(t) = y_2(t)[-d(t) + \dfrac{f(t)y_1^p(t-\tau_2)}{m(t)y_2^p(t-\tau_2) + y_1^p(t-\tau_2)}]. \end{cases} \tag{4}$$

initial conditions also is (3), where $a(t), b(t), c(t), m(t), d(t)$ and $f(t)$ are strictly positive continuous ω-periodic functions. $p \geq 1$ is a real number. Our purpose in this paper is, by using the

continuation theorem of coincidence degree theory, to establish the existence conditions of at least one positive ω-periodic solution of system(4).

Main results

For convenience of use, we introduce the continuation theory [9] as follows.

Lemma2.1 Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume

(a) for each $\lambda \in (0,1)$, $x \in \partial\Omega \bigcap \mathrm{Dom}L$, $Lx \neq \lambda Nx$;

(b) for each $x \in \partial\Omega \bigcap \mathrm{Ker}L$, $QNx \neq 0$;

(c) $\deg\{JQN, \Omega \bigcap \mathrm{Ker}L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \bigcap \mathrm{Dom}L$.

Lemma2.2 $R_+^2 = \{(y_1, y_2) \mid y_i > 0, i = 1,2\}$ is positive invariant set of system (4).

Proof From equation (4),we can obtain

$$y_1(t) = y_1(0)\exp\{\int_0^t [a(s) - b(s)y_1(s-\tau_1) - \frac{c(s)y_2^p(s)}{m(s)y_2^p(s) + y_1^p(s)}]\mathrm{d}s\} > 0, \text{ for } y_1(0) > 0,$$

$$y_2(t) = y_2(0)\exp\{\int_0^t [-d(s) + \frac{f(s)y_1^p(s-\tau_2)}{m(s)y_2^p(s-\tau_2) + y_1^p(s-\tau_2)}]\mathrm{d}s\} > 0, \text{ for } y_2(0) > 0.$$

The proof is complete.

In what follows we shall use the notations

$$\bar{f} = \frac{1}{\omega}\int_0^\omega f(t)\mathrm{d}t, \quad f^l = \min_{t\in[0,\omega]} f(t), \quad f^u = \max_{t\in[0,\omega]} f(t)$$

where f is a continuous ω-periodic function. Our main result in this paper is the following theorem about the existence of a positive ω-periodic solution of system (4).

Theorem2.1 Assume the following conditions are satisfied

(H_1) $f(t) - d(t) > 0$,

(H_2) $\overline{ma - c} > 0$.

Then system (4) has at least one positive ω-periodic solution.

Proof Let

$$x_1(t) = \ln y_1(t), \quad x_2(t) = \ln y_2(t). \tag{5}$$

On substituting (5) into (4), we rewrite (4) in the form

$$\begin{aligned} x_1'(t) &= a(t) - b(t)\exp(x_1(t-\tau_1)) - \frac{c(t)\{\exp(x_2(t))\}^p}{m(t)\{\exp(x_2(t))\}^p + \{\exp(x_1(t))\}^p}, \\ x_2'(t) &= -d(t) + \frac{f(t)\{\exp(x_1(t-\tau_2))\}^p}{m(t)\{\exp(x_2(t-\tau_2))\}^p + \exp(x_1(t-\tau_2))\}^p}. \end{aligned} \tag{6}$$

So to complete the proof, it suffices to show that system (6) has at least one ω-periodic solution. Take

$$X = Y = \{(x_1(t), x_2(t))^T \in C(\mathrm{R}, \mathrm{R}^2) : x_i(t+\omega) = x_i(t), i = 1,2\},$$

$$\left\|(x_1(t), x_2(t))^T\right\| = \max_{t\in[0,\omega]}(|x_1(t)|, |x_2(t)|).$$

Then X and Y are Banach spaces with the above norm $\|\bullet\|$, Set

$$L : \mathrm{Dom}L \subset X \to Y,$$

$$L(x_1(t), x_2(t))^T = (x_1'(t), x_2'(t))^T,$$

where $\mathrm{Dom}L = \{(x_1(t), x_2(t))^T \in C^1(\mathrm{R}, \mathrm{R}^2)\}$, and $N : X \to Y$,

$$N\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a(t) - b(t)\exp(x_1(t-\tau_1)) - \dfrac{c(t)\{\exp(x_2(t))\}^p}{m(t)\{\exp(x_2(t))\}^p + \{\exp(x_1(t))\}^p} \\ -d(t) + \dfrac{f(t)\{\exp(x_1(t))\}^p}{m(t)\{\exp(x_2(t))\}^p + \{\exp(x_1(t))\}^p} \end{bmatrix}.$$

With these notations system (6) can be written in the form $Lx = Nx, x \in X$.

Obviously, $\mathrm{Ker}L = \mathrm{R}^2$, $\mathrm{Im}\,L = \{(x_1(t), x_2(t))^T \in X : \int_0^\omega x_i(t)\mathrm{d}t = 0, i = 1,2\}$ is closed in Y, and $\dim \mathrm{Ker}L = \mathrm{co}\dim \mathrm{Im}\,L = 2$. Therefore L is a Fredholm mapping of index zero. Now define two projectors $P : X \to X$, and $Q : Y \to Y$ as

$$P\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = Q\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in X = Y.$$

Then P and Q are continuous projectors such that $\mathrm{Im}\,P = \mathrm{Ker}L, \mathrm{Ker}Q = \mathrm{Im}\,L = \mathrm{Im}(I - Q)$. We select J, the isomorphism of $\mathrm{Im}\,Q$ onto $\mathrm{Ker}L$ as identity map. Furthermore through an easy computation we find that the inverse K_P of L_P has the form

$$K_P : \mathrm{Im}\,L \to \mathrm{Dom}L \bigcap \mathrm{Ker}P,$$

$$K_P(y) = \int_0^t y(s)ds - \frac{1}{\omega}\int_0^\omega \mathrm{d}t \int_0^t y(s)\mathrm{d}s.$$

Then $QN : X \to Y$ and $K_P(I-Q)N : X \to X$ read

$$QNx = \begin{bmatrix} \dfrac{1}{\omega}\int_0^\omega [a(t) - b(t)\exp(x_1(t-\tau_1)) - \dfrac{c(t)\{\exp(x_2(t))\}^p}{m(t)\{\exp(x_2(t)\}^p + \{\exp(x_1(t))\}^p}]\mathrm{d}t \\ \dfrac{1}{\omega}\int_0^\omega [-d(t) + \dfrac{f(t)\{\exp(x_1(t-\tau_2))\}^p}{m(t)\{\exp(x_2(t-\tau_2))\}^p + \{\exp(x_1(t-\tau_2))\}^p}]\mathrm{d}t \end{bmatrix},$$

$$K_P(I-Q)Nx = \int_0^t Nx(s)ds - \frac{1}{\omega}\int_0^\omega \mathrm{d}t \int_0^t Nx(s)\mathrm{d}s - (\frac{1}{\omega} - \frac{1}{2})\int_0^\omega Nx(s)\mathrm{d}s.$$

Clearly, QN and $K_P(I-Q)N$ are continuous. By using the Arzela-Ascoli theorem, it is not difficult to prove that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Therefore N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$. Corresponding to the operator equation $Lx = \lambda Nx, \lambda \in (0,1)$, we have

$$\begin{aligned} x_1'(t) &= \lambda[a(t) - b(t)\exp(x_1(t-\tau_1)) - \frac{c(t)\{\exp(x_2(t))\}^p}{m(t)\{\exp(x_2(t))\}^p + \{\exp(x_1(t))\}^p}], \\ x_2'(t) &= \lambda[-d(t) + \frac{f(t)\{\exp(x_1(t-\tau_2))\}^p}{m(t)\{\exp(x_2(t-\tau_2))\}^p + \exp(x_1(t-\tau_2))\}^p}]. \end{aligned} \tag{7}$$

Suppose that $(x_1(t), x_2(t))^T \in X$ is a solution of (7) for a certain $\lambda \in (0,1)$. Integrating (7) over the interval $[0, \omega]$, we obtain

$$\int_0^\omega \lambda[a(t) - b(t)\exp(x_1(t-\tau_1)) - \frac{c(t)\{\exp(x_2(t))\}^p}{m(t)\{\exp(x_2(t))\}^p + \{\exp(x_1(t))\}^p}]\mathrm{d}t = \int_0^\omega x_1'(t)\mathrm{d}t = 0,$$

$$\int_0^\omega \lambda[-d(t) + \frac{f(t)\{\exp(x_1(t-\tau_2))\}^p}{m(t)\{\exp(x_2(t-\tau_2))\}^p + \exp(x_1(t-\tau_2))\}^p}]\mathrm{d}t = \int_0^\omega x_2'(t)\mathrm{d}t = 0.$$

That is

$$\begin{aligned} &\int_0^\omega [b(t)\exp(x_1(t-\tau_1)) + \frac{c(t)\{\exp(x_2(t))\}^p}{m(t)\{\exp(x_2(t))\}^p + \{\exp(x_1(t))\}^p}]\mathrm{d}t = \int_0^\omega a(t)\mathrm{d}t = \bar{a}\omega, \\ &\int_0^\omega [\frac{f(t)\{\exp(x_1(t-\tau_2))\}^p}{m(t)\{\exp(x_2(t-\tau_2))\}^p + \exp(x_1(t-\tau_2))\}^p}]\mathrm{d}t = \int_0^\omega d(t)\mathrm{d}t = \bar{d}\omega. \end{aligned} \tag{8}$$

It follows from (7) and (8) that

$$\int_0^{\omega} | x_1'(t) | \mathrm{d}t \le \int_0^{\omega} [a(t) + b(t)\exp(x_1(t-\tau_1)) + \frac{c(t)\{\exp)x_2(t))\}^p}{m(t)\{\exp(x_2(t))\}^p + \{\exp(x_1(t))\}^p}]\mathrm{d}t$$

$$= 2\bar{a}\omega, \tag{9}$$

$$\int_0^{\omega} | x_2'(t) | \mathrm{d}t \le \int_0^{\omega} [d(t) + \frac{f(t)\{\exp(x_1(t-\tau_2))\}^p}{m(t)\{\exp(x_2(t-\tau_2))\}^p + \{\exp(x_1(t-\tau_2))\}^p}]\mathrm{d}t$$

$$= 2\bar{d}\omega. \tag{10}$$

Since $(x_1(t), x_2(t))^T \in X$, there exist $\xi_i, \eta_i \in [0,\omega]$ such that

$$\begin{aligned} x_i(\xi_i) &= \min_{t\in[0,\omega]} x_i(t), \\ x_i(\eta_i) &= \max_{t\in[0,\omega]} x_i(t), \end{aligned} \quad i = 1,2. \tag{11}$$

Then from (8) we have

$$\exp(x_1(\xi_1))\int_0^{\omega} b(t)\mathrm{d}t \le \int_0^{\omega} b(t)\exp(x_1(t-\tau_1))\mathrm{d}t < \bar{a}\omega,$$

which implies

$$x_1(\xi_1) < \ln\frac{\bar{a}}{\bar{b}} := A_1.$$

This, together with (9), leads to

$$x_1(t) \le x_1(\xi_1) + \int_0^{\omega} | x'(t) | dt < A_1 + 2\bar{a}\omega. \tag{12}$$

From (8), we also have

$$\exp(x_1(\eta_1))\int_0^{\omega} b(t)\mathrm{d}t \ge \int_0^{\omega} b(t)\exp(x_1(t-\tau_1))\mathrm{d}t$$

$$= \bar{a}\omega - \int_0^{\omega} \frac{c(t)\{\exp(x_2(t))\}^p}{m(t)\{\exp(x_2(t))\}^p + \{\exp(x_1(t))\}^p}\mathrm{d}t \ge \frac{\overline{ma-c}}{m^u}\omega,$$

which implies

$$x_1(\eta_1) > \ln\frac{\overline{ma-c}}{\bar{b}m^u} := A_2.$$

This, together with (9), leads to

$$x_1(t) \ge x_1(\eta_1) - \int_0^{\omega} | x'(t) | dt > A_2 - 2\bar{a}\omega. \tag{13}$$

It follows from (12) and (13) that

$$\max_{t\in[0,\omega]} | x_1(t) | < \max\{| A_1 |, | A_2 |\} + 2\bar{a}\omega := R_1. \tag{14}$$

From (11) and the second equation of (7), we can derive

$$-d(\xi_2) + \frac{f(t)\{\exp(x_1(\xi_2-\tau_2))\}^p}{m(\xi_2)\{\exp(x_2(\xi_2-\tau_2))\}^p + \{\exp(x_1(\xi_2-\tau_2))\}^p} = 0,$$

which is equivalent to

$$d(\xi_2)m(\xi_2)\{\exp(x_2(\xi_2-\tau_2))\}^p = [f(\xi_2) - d(\xi_2)]\{\exp(x_1(\xi_2-\tau_2))\}^p \quad . \tag{15}$$

Set $\xi_2 - \tau_2 = t_0 + n\omega$, where $t_0 \in [0,\omega]$, n is an integer. Since periodicity of $(x_1(t), x_2(t))^T$, we get

$$\begin{aligned} x_1(\xi_2-\tau_2) &= x_1(t_0), \\ x_2(\xi_2-\tau_2) &= x_2(t_0). \end{aligned} \tag{16}$$

It follows from (14), (15) and (16) that

$$| x_2(t_0) | \le \max\{\frac{1}{p} | \ln[\frac{f-d}{md}]^u |, \frac{1}{p} | \ln[\frac{f-d}{md}]^l |\} + R_1 := A_3.$$

This, together with (10), leads to

$$\max_{t\in[0,\omega]} | x_2(t) | \le | x_2(t_0) | + \int_0^{\omega} | x_2'(t) | dt < A_3 + 2\bar{d}\omega := R_2. \tag{17}$$

On the other hand, let's consider algegraic equation

$$\begin{aligned}&\bar{a}-\bar{b}\exp(x_1)-\frac{\mu\{\exp(x_2)\}^p}{\omega}\int_0^\omega\frac{c(t)}{m(t)\{\exp(x_2)\}^p+\{\exp(x_1)\}^p}\mathrm{d}t=0,\\&-\bar{d}+\frac{\{\exp(x_1)\}^p}{\omega}\int_0^\omega\frac{f(t)}{m(t)\{\exp(x_2)\}^p+\{\exp(x_1)\}^p}dt=0,\end{aligned}\tag{18}$$

for $(x_1,x_2)^T\in\mathrm{R}^2$, where $\mu\in[0,1]$. Similar arguments in (14) and (17) show that

$$|x_1|\le\max\{|A_1|,|A_2|\}\le R_1\quad,\tag{19}$$

$$|x_2|\le\max\{\frac{1}{p}|\ln[\frac{f-d}{md}]^u|,\frac{1}{p}|\ln[\frac{f-d}{md}]^l|\}+R_1\le R_2.\tag{20}$$

Clearly, R_1 and R_2 are independent of λ. Take $M=2R_1+2R_2$, and define

$$\Omega=\{(x_1(t),x_2(t))^T\in X:\|(x_1(t),x_2(t))^T\|<M\}.$$

It is clear that Ω verifies the conditions (a) and (b) in Lemma2.1.

In order to verify the condition (c) in Lemma2.1, we define

$$\phi:\mathrm{Dom}L\cap\mathrm{Ker}L\times[0,1]\to X$$

by

$$\begin{aligned}\phi(x_1,x_2,\mu)=&\begin{bmatrix}\bar{a}-\bar{b}\exp(x_1)\\-\bar{d}+\frac{\{\exp(x_1)\}^p}{\omega}\int_0^\omega\frac{f(t)}{m(t)\{\exp(x_2)\}^p+\{\exp(x_1)\}^p}\mathrm{d}t\end{bmatrix}\\&+\mu\begin{bmatrix}-\frac{\{\exp(x_2)\}^p}{\omega}\int_0^\omega\frac{c(t)}{m(t)\{\exp(x_2)\}^p+\{\exp(x_1)\}^p}\mathrm{d}t\\0\end{bmatrix}\end{aligned}$$

where $\mu\in[0,1]$ is a parameter. If $\phi(x_1,x_2,\mu)=0$, then $(x_1,x_2)^T$ is the constant solution of the equation (18). From (19) and (20) we know

$$\phi(x_1,x_2,\mu)\ne0\quad\text{on}\quad\partial\Omega\cap\mathrm{Ker}L.$$

So due to homotopy invariance theorem of topology degree we have

$$\begin{aligned}&\deg(JQN(x_1,x_2)^T,\Omega\cap\mathrm{Ker}L,(0,0)^T)\\&=\deg(\phi(x_1,x_2,1),\Omega\cap\mathrm{Ker}L,(0,0)^T)\\&=\deg(\phi(x_1,x_2,0),\Omega\cap\mathrm{Ker}L,(0,0)^T)\\&=\deg((\bar{a}-\bar{b}\exp(x_1),-\bar{d}+\\&\frac{\{\exp(x_1)\}^p}{\omega}\int_0^\omega\frac{f(t)}{m(t)\{\exp(x_2)\}^p+\{\exp(x_1)\}^p}dt)^T,\Omega\cap\mathrm{Ker}L,(0,0)^T).\end{aligned}$$

It is easy to see that the following algebraic equation

$$\begin{aligned}&\bar{a}-\bar{b}\exp(x_1)=0,\\&-\bar{d}+\frac{\{\exp(x_1)\}^p}{\omega}\int_0^\omega\frac{f(t)}{m(t)\{\exp(x_2)\}^p+\{\exp(x_1)\}^p}\mathrm{d}t=0\end{aligned}$$

has a unique solution $(x_1^*,x_2^*)^T\in\mathrm{R}^2$. Thus

$$\begin{aligned}&\deg(JQN(x_1,x_2)^T,\Omega\cap\mathrm{Ker}L,(0,0)^T)\\&=\mathrm{sgn}\begin{bmatrix}-\bar{b}\exp(x_1^*) & 0\\\bar{d}-\frac{\{\exp(2x_1^*)\}^p}{\omega}\int_0^\omega\frac{f(t)\mathrm{d}t}{[m(t)\{\exp(x_2^*)\}^p+\{\exp(x_1^*)\}^p]^2} & -\frac{\{\exp(x_1^*+x_2^*)\}^p}{\omega}\int_0^\omega\frac{m(t)f(t)\mathrm{d}t}{[m(t)\{\exp(x_2^*)\}^p+\{\exp(x_1^*)\}^p]^2}\end{bmatrix}=1\ne0.\end{aligned}$$

By now we have proved that Ω satisfies all conditions in Lemma2.1. Hence (6) has at least one ω-periodic solution. Accordingly, system (4) has at least one positive ω-periodic solution. The proof is complete.

Remark If $p = 1$, the result becomes Theorem 1.1.

Conclusion

In this paper, we discussed the existence of positive periodic solution to a Ratio-Dependent Predator- Prey System with Time Delays. Sufficient conditions are obtained for the existence of positive periodic solution to system (4). Our work generalizes the known result.

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