Research on the Ultimate Value of Compound Function in Higher Mathematics

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Abstract. The problem of ultimate value of compound function is very complex, especially the research on judging whether the existence of ultimate value of compound function under the given conditions and how is the exact conditions that can affect the existence of ultimate value is not enough, but this is the most attractive research questions for math researchers. This paper discuss the ultimate value of compound functions regarding to its existence possibility in two dimensions of compound functions theory and three layers of ultimate theory, so as to effectively solve the computation problems of ultimate value of compound functions.

Introduction

The problem of ultimate value of compound function $f[g(x)]$ is very complex, because it should be firstly judged whether the $\lim f[g(x)]$ is existing or not, especially by using what kind of method to judge the existence of the ultimate value of the compound function, secondly is how to calculate the $\lim f[g(x)]$ .so that, separate the compound function into two layers.

（1）the first layer: set $u = g(x)$, and further considerate the existence of $\lim u = \lim_{x \to x_0} g(x)$

（2）the second layer: set $y = f(u)$, and further considerate the existence of $\lim y = \lim f(u)$.

Based on the above settings and foundations, this paper will discuss the existence of $\lim f[g(x)]$ and the computation problems with the given conditions which growing from weak to strong.

Under the situation of weak Conditions

Suppose the ultimate value of $\lim_{x \to x_0} u = \lim_{x \to x_0} g(x) = u_0$ is existing. Andthe ultimate value of $\lim_{u \to u_0} = \lim_{u \to u_0} f(u) = A$ is also existing, then the question is: 1）it the ultimate value of $\lim f[g(x)]$ existing or not?；（2）under the situation of the ultimate, value of $\lim f[g(x)]$ is existing,how to calculate the value of $\lim f[g(x)]$ .The answer is: it is very easy to solve the second question, which the ultimate value is $\lim f[g(x)] = \lim_{u \to u_0} f(u) = A$ .The reason is very simple, because the existing of $\lim f[g(x)]$ is true,so that it only need to concern when x is belongs to alignment $x_1, x_2, x_3, \cdots$ and x has the tendency to become x_0 ,correspondingly the u belongs to alignment $u_1, u_2, u_3, \cdots$ and u has the tendency to become u_0 , obviously, this is the solution. It is equivalent to when $\forall \varepsilon > 0, \exists \delta > 0,$ and $0 < |x - x_0| < \delta$,there is $|u - u_0| < \eta$ （$\eta > 0$）,so as to get the result of $|f[g(x)] - A| < \varepsilon$ is validate.

（1）So now, the critical problem is to solve question 1. we look at the first sample, set

$y=[1-u^8]$,represent the integrate part of $(1-u^8)$,and further set $u=x\sin\dfrac{1}{x}$,obviously, there is

$\lim\limits_{x\to 0}u=\lim\limits_{x\to 0}x\sin\dfrac{1}{x}=0$, $\lim\limits_{u\to 0}y=\lim\limits_{u\to 0}[1-u^8]=0$ however, when $x\to 0$, $u=x\sin\dfrac{1}{x}$,it can take

$x=\dfrac{1}{2k\pi}(k=1,2,3,\cdots)$,then, $u=x\sin\dfrac{1}{x}$ can get zero by infinite times, but when $u=0$, $y=1$,so

that the function $y=[1-u^8]=[1-(x\sin\dfrac{1}{x})^8]$ when $x=\dfrac{1}{2k\pi}$,then $y=1$,1,it means the function has

sub-alignments tends to 1,on the other side,when $0<\left|x\sin\dfrac{1}{x}\right|<1$,then $y=0$,so that the ultimate

value of $\lim\limits_{u\to 0}y=\lim\limits_{x\to 0}[1-(x\sin\dfrac{1}{x})^8]=0$, So that the function must has another sub-alignments , and its

tend to value0. Based on the interval column compactness theorem, the necessary and sufficient

conditions of alignment $a_n\to a$ is the sub-alignment of a_n , which is $a_{n_k}\to a$ is knowable. $\lim y$ is

not exist,Then the answer for question 1 is negative.

Critical Analysis： although theultimate value of $\lim\limits_{x\to x_0}u=\lim\limits_{x\to x_0}g(x)=u_0$ and $\lim\limits_{u\to u_0}=\lim\limits_{u\to u_0}f(u)=A$

are both existing，but in the process of replace the $u=g(x)$, when $x\to 0$, it may change u that

within a valid range which makes the ultimate value of $\lim\limits_{u\to u_0}f(u)=A$ is existing during the process

of $u\to u_0$, it is shown in the upper case.it is in the replacement of $u=x\sin\dfrac{1}{x}$, when $x\to 0$, u can

get infinite times to be 0，so that it can change the parameter range that makes $\lim y$ non-existing.

Based on the above analysis, we found, in order to make sure question (1) has the definite
answer, it has two ways of solutions, first is complete the additional conditions, second is change
the parameter range of. So that, the following questions is proposed and then sum up the regulations

Set $\lim\limits_{x\to x_0}u=\lim\limits_{x\to x_0}g(x)=u_0$, $\lim\limits_{u\to u_0}f(u)=L$,in which conditions then the ultimate value is

$$\lim\limits_{x\to x_0}f[g(x)]=L?$$

from the definition of ultimate value, set $\varepsilon>0,$ then η is existing, when

$0<\left|u-u_0\right|<\left|g(x)-u_0\right|<\eta$, then

$$\left|f(u)-L\right|=\left|f(g(x))-L\right|<\varepsilon \quad \text{is true} \qquad （3）$$

because $\lim\limits_{x\to x_0}g(x)=u_0$, so $\varepsilon>0,$ then $\delta>0$ existing, makes when $0<\left|x-x_0\right|<\delta$, there is

$$\left|g(x)-u_0\right|<\eta \qquad （4）$$

in order to proof $\lim\limits_{x\to x_0}f[g(x)]=L$, is equal to when $0<\left|x-x_0\right|<\delta$, there is

$$\left|f(g(x))-L\right|<\varepsilon \quad \text{is true}$$

it is to proof from（4）to（3），So that，in（3），set $u\neq u_0$, if $u=u_0$, then the in-equivalent

$$\left|f(u)-L\right|<\varepsilon$$

may not true（for example the situation of $u=0$ ），based on the upon example, there is exist the
x to make the in-equivalent

$$0 < |x - x_0| < \delta$$ is true

, then $u = g(x) = u_0$ (for example the $x = \dfrac{1}{2k\pi}, k = 1,2,3,\cdots, \ u = x\sin\dfrac{1}{x} = 0$),but $|f(u) - L| < \varepsilon$ is not reasonable., so that, under the conditions for the given problems it is hard to guarantee the (3) can be proofed from (4),: so that the additional conditions is needed.,

I、in（4）$g(x) = u_0$ there is no x to make it true

II、when $u = u_0$, $|f(u) - L| < \varepsilon$ is true,

Based on the above analysis, the discipline for calculating the ultimate value of compound function is:

1 set $\lim\limits_{x \to x_0} u = \lim\limits_{x \to x_0} g(x) = u_0$, $\lim\limits_{u \to u_0} f(u) = L$, when $x = x_0$ （or exclude $x = x_0$）there is no x makes $g(x) = u_0$,then

$$\lim_{x \to x_0} f[g(x)] = \lim_{u \to u_0} f(u) = L$$

1 based on $\lim\limits_{x \to \frac{\pi}{2}} u = \lim\limits_{x \to \frac{\pi}{2}} \cos x = 0$,and $\lim\limits_{u \to 0} \dfrac{\sin u}{u} = 1$,in this example,when $x \to \dfrac{\pi}{2}$,there is no x to make $\cos x = 0$,based on rule1there is

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin(\cos x)}{\cos x} = \lim_{u \to 0} \frac{\sin u}{u} = 1$$

Under the Strong conditions

in the disciple 1 the conditions is $\lim\limits_{x \to x_0} u = \lim\limits_{x \to x_0} g(x) = u_0$,based on $\lim\limits_{x \to u_0} f(u) = L$,set $f(u_0) = L$,because $f(u_0) = L$ effectively constrains the range of the variable u ,or can say it exclude the situation when $x \to x_0$, during $u \to u_0$ it may change the range of u ,which makes the $\lim\limits_{u \to u_0} f(u) = A$ is existing,,: based on the above analysis:

set $\lim\limits_{x \to x_0} u = \lim\limits_{x \to x_0} g(x) = u_0$, $\lim\limits_{x \to u_0} f(u) = L$,and $f(u_0) = L$,then

$$\lim_{x \to x_0} f(g(x)) = \lim_{u \to u_0} f(u)$$

example2 $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = e$, $y = \ln(1 + x)^{\frac{1}{x}}$,because $y = \ln u$ and $f(u_0) = L$,set $\ln u\big|_{u=e} = \ln e$,based on rule2

$$\lim_{x \to 0} \ln(1 + x)^{\frac{1}{x}} = \ln[\lim_{x \to o}(1 + x)^{\frac{1}{x}}] = \ln e = 1$$

example 3 $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = y'$, $\lim\limits_{x \to x_0} u = \lim\limits_{x \to x_0} g(x) = u_0$, $\lim\limits_{x \to u_0} f(u) = L$,and $f(u_0) = L$,based on rule 2

$$\lim_{\Delta x \to 0} \sqrt{1 + (\frac{\Delta y}{\Delta x})^2} = \sqrt{1 + (\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x})^2} = \sqrt{1 + (y')^2}$$

So that, when there is a compound function $f(g(x))$,and $u = g(x)$, based on the character of continuous functions, it is very ease to get:

$f(u)$, $u = u_0$ is not continuous,or $f(u_0)$ no definition,or $f(u_0) \neq \lim\limits_{u \to u_0} f(u)$;

rule 2 is suitable for $f(u)$,when $u = u_0$,the $g(x)$ is continuous at $x = x_0$ has no relation with question,only makes $\lim\limits_{x \to x_0} g(x)$ existing is fine.

III Condition is very Strong

Under the rule2 $\lim\limits_{x \to x_0} u = \lim\limits_{x \to x_0} g(x) = u_0$ $\lim\limits_{x \to u_0} f(u) = L$,and based on $f(u_0) = L$ further add the conditon $u_0 = g(x_0)$,so we get rule 3:

Rule 3, set $\lim\limits_{x \to x_0} u = \lim\limits_{x \to x_0} g(x) = g(x_0) = u_0$ $\lim\limits_{u \to u_0} f(u) = f(u_0) = A$ there is

$$\lim\limits_{x \to x_0} f(g(x)) = A$$

It needs cautions that, because the primary functions are all continuous functions within it domain, so that, when discussion the ultimate value of primary compound functions, the discipline 3 can always be used for solving such problems, but this doesn't mean that all the ultimate value of compound functions can be solved by using discipline 3, the example 2 and 3 are both this kinds of situations, while we can not considerate that.

$$\lim\limits_{x \to 0} \ln(1 + x)^{\frac{1}{x}} = \ln[\lim\limits_{x \to o}(1 + x)^{\frac{1}{x}}] = \ln e = 1$$

is because of the continuality of $\ln u$,because $\ln u$ is the continuous function of u , $\lim\limits_{u \to e} \ln u = \ln e$,but $\ln(1+x)^{\frac{1}{x}}$ and $g(x) = (1+x)^{\frac{1}{x}}$ are not continuous at $x = 0$, so that the disciple 3 can not be applied here, especially it can be point out that, the inverse function is a ultimate value problems of compound functions.

Set $g(x) = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(x_0)}{x \to x_0}$, $f(u) = \dfrac{1}{u}$,

Obviously $u_0 \neq 0$ there is $\lim\limits_{u \to u_0} f(u) = \lim\limits_{u \to u_0} \dfrac{1}{u} = \dfrac{1}{u_0}$,

But when $\lim\limits_{x \to x_0} g(x) = f'(x_0) = u_0$, is there existing $\lim\limits_{x \to x_0} f[g(x)] = \lim\limits_{u \to u_0} f(u) = \dfrac{1}{u_0}$ is a problem.

so that, it must based on the conditions of getting inverse functions : first, the inverse function is existing ,then the mapping of its original functions must be one to one mapping,so that when $x \neq x_0$, $y \neq y_0$, Second, because $u_0 \neq 0$,and $f(u_0) = \dfrac{1}{u_0}$ is existing,this exclude when $x \to x_0$, during $u \to u_0$ the range of u make change the range value of $\lim\limits_{u \to u_0} f(u) = A$, so based on rule 1,

$$\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{\lim\limits_{\Delta y \to 0} \dfrac{\Delta x}{\Delta y}} = \frac{1}{x'(y)} \qquad x'(y) \neq 0.$$

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