Positive solutions of the fourth-order boundary value problem with dependence on the first order derivative

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Abstract: In this paper, By the use of a new fixed point theorem and the Green function. The existence of at least one positive solutions for the fourth-order boundary value problem with the first order derivative

$$\begin{cases} u^{(4)}(t) + Au''(t) = \lambda f(t, u(t), u'(t)) \quad 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

is considered, where f is a nonnegative continuous function and $\lambda > 0, 0 < A < \pi^2$.

1. Introduction

Recently, there has been much attention focused on the question of positive solution of fourth-order differential equation with one or two parameters. For example, astronomy, biology, physics, chemical engineering and information science and other fields. So, the fourth-order boundary value problems has very important in real life applications, see for example [1-4, 6-9].

Li [6] investigated the existence of positive solutions for the fourth-order boundaryvalue problem. All the above works were done under the assumption that the first order derivative u' is not involved explicitly in the nonlinear term f . In this paper, we are concerned with the existence of positive solutions for the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) + Au''(t) = \lambda f(t, u(t), u'(t)) \quad 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{1}$$

The following conditions are satisfied throughout this paper:

(H_1) $\lambda > 0, 0 < A < \pi^2$;

(H_2) $f : [0,1] \times [0, \infty) \times R \to [0, \infty)$ is continuous.

2. The preliminary lemmas

Suppose $Y = C[0,1]$ be the Banach space equipped with the norm $\|u\|_0 = \max_{t \in [0,1]} |u(t)|$.

Let λ_1, λ_2 be the roots of the polynomial $P(\lambda) = \lambda^2 + A\lambda$, namely, $\lambda_1 = 0, \lambda_2 = -A$. By ($H_1$) it is easy to see that $-\pi^2 < \lambda_2 < 0$.

Let $G_i(t,s)(i=1,2)$ be the Green ' s function of the linear boundary value problem: $-u''(t) + \lambda_i u(t) = 0, u(0) = u(1) = 0$. Then, carefully calculation yield:

$$G_1(t,s) = \begin{cases} s(1-t), 0 \le s \le t \le 1 \\ t(1-s), 0 \le t \le s \le 1 \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{\sin\sqrt{A}s \sin\sqrt{A}(1-t)}{\sqrt{A}\sin\sqrt{A}}, 0 \le s \le t \le 1 \\ \dfrac{\sin\sqrt{A}t \sin\sqrt{A}(1-s)}{\sqrt{A}\sin\sqrt{A}}, 0 \le t \le s \le 1 \end{cases}$$

Lemma 2.1: Suppose (H_1) (H_2) hold. Then for any $g(t) \in C[0,1]$, BVP

$$\begin{cases} u^{(4)}(t) + Au''(t) = g(t), 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{2}$$

the unique solution $u(t) = \int_0^1 \int_0^1 G_1(t,s) G_2(s,\tau) g(\tau) \mathrm{d}\tau \mathrm{d}s$. (3)

where

$$G_1(t,s) = \begin{cases} s(1-t), 0 \le s \le t \le 1 \\ t(1-s), 0 \le t \le s \le 1 \end{cases},$$

$$G_2(s,\tau) = \begin{cases} \dfrac{\sin\sqrt{A}\tau \sin\sqrt{A}(1-s)}{\sqrt{A}\sin\sqrt{A}}, 0 \le \tau \le s \le 1 \\ \dfrac{\sin\sqrt{A}s \sin\sqrt{A}(1-\tau)}{\sqrt{A}\sin\sqrt{A}}, 0 \le s \le \tau \le 1 \end{cases}$$

Lemma 2.2[5] : Assume (H_1) (H_2) hold. Then one has:

(i) $G_i(t,s) \ge 0, \forall t,s \in [0,1]$;

(ii) $G_i(t,s) \le C_i G_i(s,s), \forall t,s \in [0,1]$;

(iii) $G_i(t,s) \ge \delta_i G_i(t,t) G_i(s,s), \forall t,s \in [0,1]$.

Where: $C_1 = 1, \delta_1 = 1; C_2 = \dfrac{1}{\sin\sqrt{A}}, \delta_2 = \sqrt{A}\sin\sqrt{A}$.

Lemma 2.3: Assume (H_1) (H_2) hold and are given as above, Then one has:

$$\min_{\frac{1}{4} \le t \le \frac{3}{4}} u(t) \ge d \|u\|_0$$

where: $d=\dfrac{\sqrt{A}\sin^2\sqrt{A}C_0G_0}{M_1}$, $C_0=\int_0^1 G_1(s,s)G_2(s,s)\mathrm{d}s$, $M_1=\int_0^1 G_1(s,s)\mathrm{d}s$, $G_0=\min_{t\in[\frac{1}{4},\frac{3}{4}]} G_1(t,t)$.

Proof: By(3)and (ii) of Lemma2.2,we get:

$$u(t)\le C_1C_2\int_0^1\int_0^1 G_1(s,s)G_2(\tau,\tau)g(\tau)\mathrm{d}\tau\mathrm{d}s\le C_1C_2M_1\int_0^1 G_2(\tau,\tau)g(\tau)\mathrm{d}\tau$$

Therefore, $\|u\|_0\le C_1C_2M_1\int_0^1 G_2(\tau,\tau)g(\tau)\mathrm{d}\tau$

By (iii)of Lemma2.2, we have:

$$u(t)\ge\delta_1\delta_2\int_0^1\int_0^1 G_1(t,t)G_1(s,s)G_2(s,s)G_2(\tau,\tau)g(\tau)\mathrm{d}\tau\mathrm{d}s$$

$$=\delta_1\delta_2C_0G_1(t,t)\int_0^1 G_2(\tau,\tau)g(\tau)\mathrm{d}\tau$$

$$\ge\frac{\delta_1\delta_2C_0}{C_1C_2M_1}G_1(t,t)\|u\|_0$$

Let $G_0=\min_{t\in[\frac{1}{4},\frac{3}{4}]} G_1(t,t)$, we have:

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t)\ge\frac{\delta_1\delta_2C_0G_0}{C_1C_2M_1}\|u\|_0$$

$$=\frac{\sqrt{A}\sin^2\sqrt{A}C_0G_0}{M_1}\|u\|_0$$

$$=d\|u\|_0$$

Theorem 2.1[10]: Let $r_2>r_1>0, L>0$ be constants and

$$\Omega_i=\{u\in X:\alpha(u)<r_i,\beta(u)<L\}, i=1,2$$

two bounded open sets in X. Set $D_i=\{u\in X:\alpha(u)=r_i,\}, i=1,2$;

Assume $T:K\to K$ is a completely continuous operator satisfying:

(A_1) $\alpha(Tu)<r_1, u\in D_1\cap K;\alpha(Tu)>r_2, u\in D_2\cap K$;

(A_2) $\beta(Tu)<L, u\in K$;

(A_3)there is $\exists p\in(\Omega_2\cap K)\setminus\{0\}$,

such that $\alpha(p)\ne 0$ and $\alpha(u+\lambda p)\ge\alpha(u)$,for all $\forall u\in K,\lambda\ge 0$.

Then T has at least one fixed point in $(\Omega_2\setminus\overline{\Omega_1})\cap K$.

3. The main results

Let $X = C^1[0,1]$ be the Banach space equipped with the norm $\|u\| = \max_{t\in[0,1]}|u(t)| + \max_{t\in[0,1]}|u^{'}(t)|$, and

$K = \left\{u \in X : u \geq 0, \min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \geq d\,\|u\|_0\right\}$ is a cone in X.

Define functionals $\alpha(u) = \max_{t\in[0,1]}|u(t)|$, $\beta(u) = \max_{t\in[0,1]}|u^{'}(t)|$, $\forall u \in X$.

then, $\|u\| \leq 2\max\{\alpha(u),\beta(u)\}$, $\alpha(\lambda u) = |\lambda|\alpha(u)$, $\beta(\lambda u) = |\lambda|\beta(u)$, $\forall u \in X, \lambda \in R$,

$\alpha(u) \leq \alpha(v), \forall u, v \in K, u \leq v$.

Assume (H_1) hold, the green's function of the problem (2) $G_i(t,s) \geq 0$. let $g(t) = 1$, we have

$$\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s = \frac{\sin\sqrt{A}(1-t)+\sin\sqrt{A}t}{A^2\sin\sqrt{A}} + \frac{t^2-t}{2A} - \frac{1}{A^2}$$

we denote:

$$M = \max_{t\in[0,1]}\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s,\quad m = \max_{t\in[\frac{1}{4},\frac{3}{4}]}\int_0^1\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(t,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s$$

$$Q = \frac{2A^2\sin\sqrt{A}}{[6\sqrt{A}-(1-\cos\sqrt{A})-3\sin\sqrt{A}]}$$

We will suppose that there are $\exists L > b > db > c > 0$, such that $f(t,u,v)$ $f(t,u,v)$

satisfies the following growth conditions:

(H_3) $f(t,u,v) < \dfrac{c}{\lambda M}, \forall (t,u,v) \in [0,1]\times[0,c]\times[-L,L]$;

(H_4) $f(t,u,v) \geq \dfrac{b}{\lambda m}, \forall (t,u,v) \in [\frac{1}{4},\frac{3}{4}]\times[db,b]\times[-L,L]$;

(H_5) $f(t,u,v) < \dfrac{L}{\lambda Q}, \forall (t,u,v) \in [0,1]\times[0,b]\times[-L,L]$.

Let

$$f^*(t,u,v) = \begin{cases} f(t,u,v), (t,u,v) \in [0,1]\times[0,b]\times(-\infty,\infty) \\ f(t,b,v), (t,u,v) \in [0,1]\times(b,\infty)\times(-\infty,\infty) \end{cases}$$

$$f_1(t,u,v) = \begin{cases} f^*(t,u,v), (t,u,v) \in [0,1]\times[0,\infty)\times[-L,L] \\ f^*(t,u,-L), (t,u,v) \in [0,1]\times[0,\infty)\times(-\infty,-L] \\ f^*(t,u,L), (t,u,v) \in [0,1]\times[0,\infty)\times[L,\infty) \end{cases}$$

Define:

$$(Tu)(t) = \lambda\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)\,f_1(\tau,u(\tau),u^{'}(\tau))\mathrm{d}\tau\mathrm{d}s \tag{4}$$

$$(Tu)^{'}(t) = \lambda\Big[\int_t^1\int_0^1 G_2(s,\tau)\,f_1(\tau,u(\tau),u^{'}(\tau))\mathrm{d}\tau\mathrm{d}s - \int_t^1\int_0^1 sG_2(s,\tau)\,f_1(\tau,u(\tau),u^{'}(\tau))\mathrm{d}\tau\mathrm{d}s\Big] \tag{5}$$

Lemma 3.1: Suppose (H_1) (H_2) hold , then $T: K \to K$ is completely continuous.

Proof: For $\forall u \in K$, by (5) and Lemma 2.2, there is $Tu \geq 0$.

so,

$$\|Tu\|_0 = \max_{t\in[0,1]}\left|\lambda\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\mathrm{d}s\right|$$
$$\le \lambda\int_0^1\int_0^1 C_1C_2G_1(s,s)G_2(\tau,\tau)f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\mathrm{d}s$$
$$\le \lambda C_1C_2M_1\int_0^1 G_2(\tau,\tau)f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau$$

we have :

$$\min_{t\in[\frac14,\frac34]}(Tu)(t) = \min_{t\in[\frac14,\frac34]}\lambda\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau ds$$
$$\ge \lambda\delta_1\delta_2\int_0^1\int_0^1 G_1(t,t)G_1(s,s)G_2(s,s)G_2(\tau,\tau)f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\mathrm{d}s$$
$$\ge \lambda\delta_1\delta_2C_0G_1(t,t)\int_0^1 G_2(\tau,\tau)f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau$$
$$\ge \lambda\delta_1\delta_2C_0G_0\int_0^1 G_2(\tau,\tau)f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau$$
$$\ge \frac{\lambda\delta_1\delta_2C_0G_0}{C_1C_2M_1}\|Tu\|_0$$
$$= d\,\|Tu\|_0$$

Therefore, we get $T(K)\subset K$.

So we can get $T(K)\subset K$ .Let $B\subset K$ is bounded, it is clear that $T(B)$ is bounded. Using $f_1, G_1(t,s), G_2(t,s)$ is continuous, we show that $T(B)$ is equicontinuous. By the Arzela-Ascoli theorem, a standard proof yields $T: K\to K$ is completely continuous.

Theorem 3.1: Suppose condition (H_1)—(H_5) hold, Then BVP (1) has at least one positive solution $u(t)$ satisfying:

$$c<\alpha(u)<b, |u'(t)|<L$$

Proof : Take $\Omega_1=\{u\in X: |u(t)|<c, |u'(t)|<L\}$, $\Omega_2=\{u\in X: |u(t)|<b, |u'(t)|<L\}$

two bounded open sets in X and $D_1=\{u\in X: \alpha(u)=c\}$, $D_2=\{u\in X:\alpha(u)=b\}$

such that $\alpha(u+\lambda p)\ge\alpha(u), \forall u\in K, \lambda\ge 0$, $\forall u\in D_1\cap K, \alpha(u)=c$,

From (H_3) we have:

$$\alpha(Tu) = \max_{t\in[0,1]}\left|\lambda\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\mathrm{d}s\right|$$
$$< \max_{t\in[0,1]}\lambda\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)\frac{c}{\lambda M}\mathrm{d}\tau\mathrm{d}s$$
$$= \frac{c}{M}\max_{t\in[0,1]}\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s$$
$$= c$$

$\forall u\in D_2\cap K, \alpha(u)=b$. From Lemma 2.3, we have $u(t)\ge d\alpha(u)=db$, $t\in[\frac14,\frac34]$,

so, from (H_4) we get:

$$\alpha(Tu) = \max_{t\in[0,1]}\left|\lambda\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\mathrm{d}s\right|$$
$$> \max_{t\in[\frac14,\frac34]}\lambda\int_0^1\int_{\frac14}^{\frac34} G_1(t,s)G_2(s,\tau)\frac{b}{\lambda m}\mathrm{d}\tau\mathrm{d}s$$

$$= \frac{b}{m} \max_{t\in[\frac{1}{4},\frac{3}{4}]} \int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(t,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s$$
$$= b$$

$\forall u \in K$, from (H_5) we get:

$$\beta(Tu) = \max_{t\in[0,1]} \left|\lambda \int_t^1 \int_0^1 G_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\mathrm{d}s - \lambda \int_0^1 \int_0^1 sG_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\mathrm{d}s\right|$$
$$< d_1\lambda \int_0^1 \int_0^1 Q_1(s,s)Q_2(s,\tau) f_1(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\mathrm{d}s$$
$$= \left(\frac{6\sqrt{A}(1-\cos\sqrt{A}) - 3A\sin\sqrt{A}}{2A^2\sin\sqrt{A}}\right) \times \frac{L}{Q} = L$$

Theorem 2.1 implies there is $u \in (\Omega_2 \setminus \overline{\Omega}_1) \cap K$, such that $u = Tu$. so, $u(t)$ is a positive solution for BVP(1), satisfying :

$c < \alpha(u) < b, \left|u'(t)\right| < L$

Thus, Theorem 3.1 is completed.

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