The Application of General Homotopy Method on Planar Four-Bar Path Synthesis

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***Abstract*—Homotopy method is a numerical continuation method that can locate all the zeros of any given function without the specifying of initial estimates. With the assistance of proper programming technique, homotopy method can be efficient and reliable. The application of homotopy method on kinematic synthesis can resolve the inherent shortcomings that most numerical methods possess. This work presents a demonstration of homotopy method on the path synthesis of planar four-bar mechanisms. The example of nine point-path synthesis is provided. The results of all real solutions are tabulated and the corresponding mechanisms are graphically displayed so comparisons can be made with past publications.**

Keywords-homotopy method; four-bar mechanisms; point-path synthesis

I. INTRODUCTION

Though numerical solutions are common to the kinematic synthesis, several inherent shortcomings need to be addressed to have the numerical methods more applicable to the kinematic synthesis. The requirement of initial estimate is a typical issue for most numerical methods as wrong estimate could lead to wrong solution or no solution at all. In addition, most numerical methods are generally not able to obtain all possible solutions.

Among the various numerical methods that are commonly used, the homotopy method [1,3,4,6-8] belongs to the family of continuation methods [2,4,5] and like all these methods they represent a way to find a solution to a problem by constructing a new problem, simpler than the original one, and then gradually deforming this simpler problem into the original one keeping track of the series of zeros that connect the solution of the simpler problem to that of the original, harder one. This nature will allow homotopy method to start the searching for solutions without any initial estimates. The greatest advantage of the homotopy method is that it offers a way to have a globally convergent method to find the zeros of all the possible solutions of any function. This characteristic will certainly be very helpful for kinematic synthesis since every possible design can be reviewed and allow better choice to be made.

II. HOMOTOPY METHOD

The homotopy process starts form the zeros p of *G(p)*=0 at t=0 as in Figure 1. And for each iteration i, a small increment Δt is added and the solution to the following system is recalculated with Newton's method.

$$H(x, t_i) = 0 \tag{1}$$

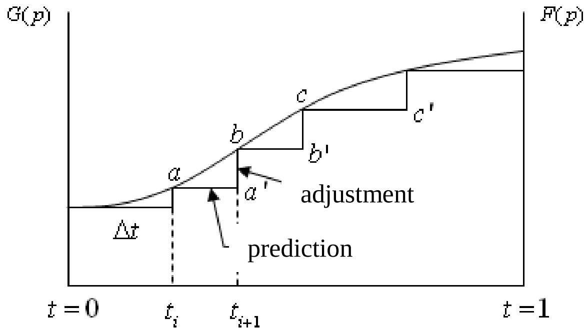

FIGURE I. CONVERGENCE OF A HOMOTOPY PATH

The solution xi to (1) will be the starting value for next interval ti+1. That is from point a to a' in Figure 1. Then the intermediate solution xi+1 is obtained with the adjustment made by Newton's method, which is from point a' to b. The course continues from point b to b' and to point c for next intermediate solution. Such a solution searching process repeats in the same manner until the homotopy path reaches t=1 where the solution to the original problem *F(x)*=0 is converged.

Tsai and Morgan [12] solved the kinematics of general six-degree-of-freedom manipulators with continuation method. Wampler, Morgan, and Sommese [13] also used the continuation method to solve the polynomial systems in kinematics. Tsai and Lu [14] applied cheater's homotopy method on the synthesis of coupler-point curve. Subbian and Flugrad [15] showed the 25 solutions of five-position four-bar path synthesis problem solved with continuation method. Wampler, Morgan, and Sommese [16] presented the complete solutions of nine-position four-bar path synthesis problem solved with polynomial continuation method. The revised constrained homotopy method was presented in [23]. Raghavan [17] presented the forward kinematics of general Stewart Platform with continuation method. Other applications have also been found in [18-26].

III. FOUR-BAR PATH SYNTHESIS

Figure 3 shows a planar four-bar mechanism with Z1 and Z3 being the initial position vectors attached to input and output links. Z2 and Z4 are the initial position vectors from the two moving pivots to the coupler point. ϕ_j, ψj, and γ_j are the angle displacements from initial position to position j of input link, output link, and coupler respectively. (δ_{jx}, δ_{jy}) is the displacement of coupler point from initial position to position j.

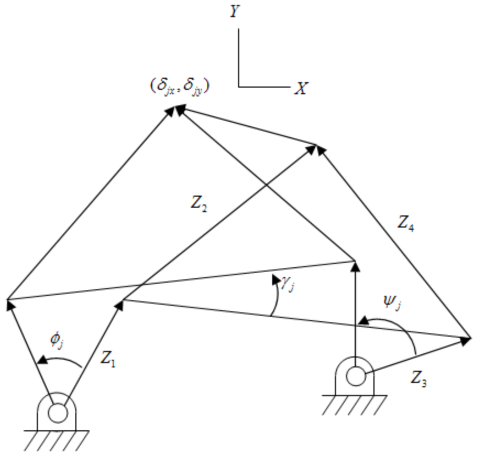

FIGURE II. A PLANAR FOUR-BAR MECHANISM

The loop closure equations for the four-bar mechanism can be written as

$$Z_1 e^{i\varphi_j} + Z_2 e^{i\gamma_j} = \delta_j + Z_1 + Z_2 \qquad (2)$$

$$Z_3 e^{i\psi_j} + Z_4 e^{i\gamma_j} = \delta_j + Z_3 + Z_4 \qquad (3)$$

Eliminating angles ϕ_j, ψ_j, and γ_j in (3) and (4) and rearranging will get

$$\begin{aligned}&(B_{1j}D_{2j} - B_{2j}D_{1j}) + 4(A_{1j}B_{2j} - A_{2j}B_{1j})\times \\ &(A_{1j}D_{2j} - A_{2j}D_{1j}) + (A_{1j}D_{2j} - A_{2j}D_{1j}) = 0\end{aligned} \qquad (4)$$

where

$$A_{1j} = Z_{1x}Z_{2y} - Z_{1y}Z_{2x} + Z_{2y}\delta_{jx} - Z_{2x}\delta_{jy}$$

$$B_{1j} = Z_{2x}{}^2 + Z_{2y}{}^2 + Z_{1x}Z_{2x} + Z_{1y}Z_{2y} + Z_{2x}\delta_{jx} + Z_{2y}\delta_{jy}$$

$$D_{1j} = 2Z_{1x}\delta_{jx} + 2Z_{1y}\delta_{jy} + \delta_{jx}{}^2 + \delta_{jy}{}^2$$

$$A_{2j} = Z_{3x}Z_{4y} - Z_{3y}Z_{4x} + Z_{4y}\delta_{jx} - Z_{4x}\delta_{jy}$$

$$B_{2j} = Z_{4x}{}^2 + Z_{4y}{}^2 + Z_{3x}Z_{4x} + Z_{3y}Z_{4y} + Z_{4x}\delta_{jx} + Z_{4y}\delta_{jy}$$

$$D_{2j} = 2Z_{3x}\delta_{jx} + 2Z_{3y}\delta_{jy} + \delta_{jx}{}^2 + \delta_{jy}{}^2$$

Equation (4) has a total of eight unknowns Z_{1x}, Z_{1y}, Z_{2x}, Z_{2y}, Z_{3x}, Z_{3y}, Z_{4x}, and Z_{4y}. Let's assume origin (0, 0) to be the initial coupler point, then the maximum number of precision points that can be determined by (4) will be nine.

IV. NINE-POINT PATH SYNTHESIS

The number of homotopy path of nine precision-point synthesis is 78=5764801. This study uses a technique called multi-homogeneous partitions of variables [9-11]. This multi-homogeneous homotopy method is able to reduce the number of paths to 286720, only five percent to its original system and is in accordance with the number proposed in [16]. Although the downsized system still requires intensive computation, the efficiency is quite acceptable. The coordinates of the nine precision-points are listed in Table 1. A total of 28 real solutions are found and listed in Table 2.

TABLE I. COORDINATES OF NINE PRECISION-POINTS

No.	δ_x	δ_y
1	0	0
2	-0.4330	-0.1602
3	-0.8092	-0.4871
4	-1.0158	-0.7936
5	-1.1578	-1.1583
6	-1.1723	-1.4218
7	-1.1058	-1.5536
8	-0.7945	-1.5852
9	-0.2712	-1.2404

There are four resulting four-bar mechanisms displayed in Figures 3, 4, 5, and 6 corresponding to solution no. 1, 9, 23, and 27 in Table 2 respectively.

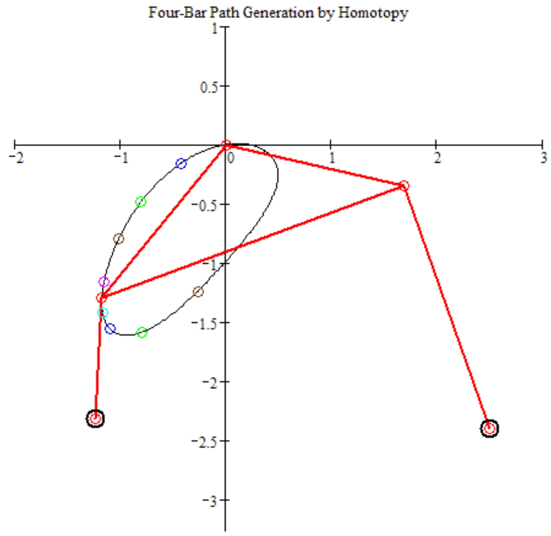

FIGURE III. MECHANISM CORRESPONDS TO SOLUTION NO 1

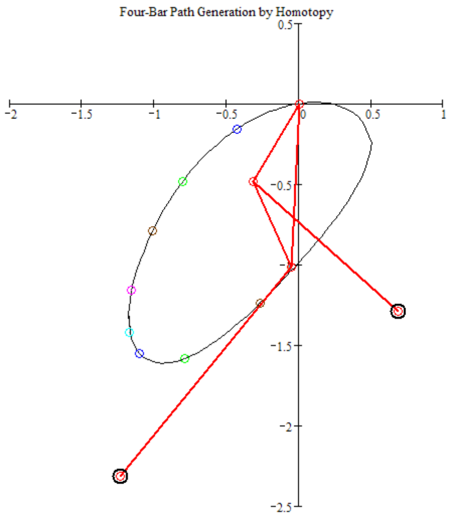

FIGURE IV. MECHANISM CORRESPONDS TO SOLUTION NO. 9

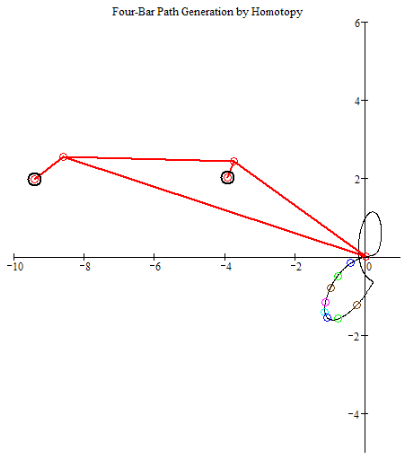

FIGURE V. MECHANISM CORRESPONDS TO SOLUTION NO. 23

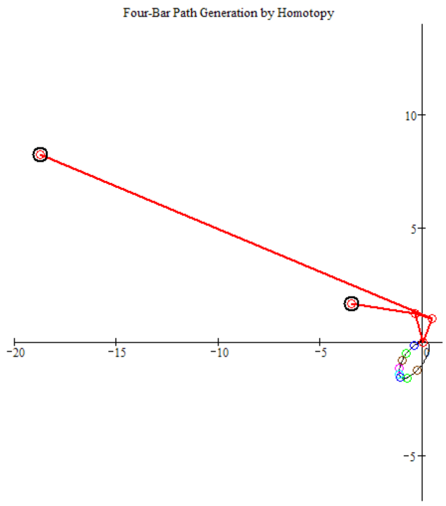

FIGURE VI. MECHANISM CORRESPONDS TO SOLUTION NO. 27

TABLE II. REAL SOLUTIONS OF NINE-POINT SYNTHESIS

No.	Z_1(x/y)	Z_2(x/y)	Z_3(x/y)	Z_4(x/y)
1	-0.8094	-1.6842	0.0530	1.1841
	2.0510	0.3515	1.0136	1.3015
2	0.3916	-0.3072	0.4859	0.2740
	0.6246	0.3306	0.5287	0.7960
3	0.8913	0.3869	0.4613	0.0615
	0.6447	0.8770	-0.2580	0.7400
4	0.8558	0.2665	0.4060	-1.8465
	1.0958	3.5867	0.3578	0.9562
5	0.1734	0.2204	0.6242	0.3944
	1.1821	0.6820	0.1928	0.3430
6	-0.8999	0.1253	1.0345	0.3264
	1.2681	0.4387	0.4841	0.9123
7	0.6184	-0.0479	1.5019	0.3617
	-1.3714	0.7606	0.9775	0.9065
8	1.2180	-0.4347	-0.3093	0.5378
	0.3104	0.9289	0.6698	-0.0269
9	-0.9967	0.3157	1.1841	0.0530
	0.8096	0.4841	1.3015	1.0136
10	0.6453	0.2460	0.1563	0.2757
	5.4799	0.7985	0.7353	0.0390
11	0.2052	0.2919	63.366	0.3547
	0.1654	0.6392	20.707	0.3783
12	1.1841	0.0530	-0.9967	0.3157
	1.3015	1.0136	0.8096	0.4841
13	0.3063	-0.1261	0.3558	0.4074
	0.5330	0.4556	0.5283	0.7414
14	0.4772	-0.0462	1.9320	0.4121
	-1.2570	0.7994	0.9248	0.8280
15	0.1630	0.7288	1.5836	-0.7814
	-0.5766	0.6267	4.2187	1.2840
16	0.4060	-1.8465	0.8558	0.2665
	0.3578	0.9562	1.0958	3.5867
17	1.5836	-0.7814	0.1630	0.7288
	4.2187	1.2840	-0.5766	0.6267
18	2.0558	0.7982	-0.1985	0.6478
	1.3935	0.1057	0.2953	0.3754
19	0.4859	0.2740	0.3916	-0.3072
	0.5287	0.7960	0.6246	0.3306
20	63.365	0.3547	0.2052	0.2919
	20.707	0.3783	0.1654	0.6392
21	0.2665	0.8558	0.1404	0.7663
	3.5867	1.0958	0.5595	-0.3836
22	-2.4059	-1.1112	0.4468	0.2373
	11.584	0.2516	-0.4824	0.7138
23	0.8186	8.6086	0.1926	3.7477
	0.5628	-2.5448	0.4151	-2.4443
24	0.3402	0.4414	0.2633	-0.0996
	0.5524	0.7251	0.5966	0.4243
25	0.2730	-1.3976	0.4273	-0.1754
	0.7425	0.5792	0.7121	3.9870
26	0.5378	-0.3092	0.2780	-0.0422
	-0.0269	0.6698	0.4112	0.5533
27	19.210	-0.4290	3.1102	0.4066
	-7.2353	-1.0220	-0.4204	-1.2683
28	0.4741	0.2651	0.4321	-0.3406
	0.5280	0.7976	0.5644	0.3519

V. CONCLUSIONS

This study presents the applications of homotopy method in the path-synthesis of four-bar mechanisms. The result of nine precision-point synthesis is presented and comparisons are made with past results.

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