

## A fully dispersive fifth order nonlinear wave model. I: Theoretical part

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**ABSTRACT:** A fully dispersive fifth order nonlinear wave propagation model for mild current, water level and depth was established theoretically considering energy and topography factors. It satisfied both the frequency dispersion and nonlinearity. By omitting the 5<sup>th</sup> order terms, the lower order model was in consistency with the former 3rd order model. This model could be simplified to mild-slope type equations for deep water, to Boussinesq type equations for shallow water and to Airy wave for very shallow water.

### INTRODUCTION

Wave is an important ocean dynamic factor. It has great influence on maritime structures safety, harbor ship berthing, sediment transport and pollutant dispersion. In the last two decades, wave propagation model developed in two main direction: 1) fully dispersive mild slope equation and its extension 2) weakly dispersive nonlinear Boussinesq-type equation and its extension. The Boussinesq-type equations are weakly dispersive nonlinear equations, suitable for shallow water and has limitations in calculating wave propagation in deep water. Though some extensions of B-Type equations improved the limitation to water depth, the situations for water depth equivalent to or larger than wave length, the results of extension models were still not satisfied. The mild slope equation is suitable for modeling linear single frequency and has limitations in shallow water wave propagation in near shore areas. Though the time-related mild-slope equation can solved the frequency problem, the band-width of spectrum was still limited. To breakthrough this limitations, the researchers proposed the mathematical models which satisfied fully dispersion and nonlinearity simultaneously. The calculation model and the applicability are yet to be discussed. Nadaoka (1994) *et al* adopted vertical dependent function to express wave velocity, and by applying Galerkin method, these vertical dependent functions were idealized combined. Those functions could be used to express any wave velocity field and finally a called fully dispersive nonlinear mathematical model was achieved. Isobe (1994) approximated the frequency related coefficient as rational function, and extend the original mild slope equations into a "time-related mild slope equation" for random waves. It can depict time evolution process of near shore short waves; Li B. (2008) adopted vertical characteristic horizontal velocity to deduce control functions. The most improvement of the new control function was that it could satisfy the dispersion relation in the range of full water, and the linear shallow gradient factor is in good agreement with Stokes theoretical solution. The authors concluded that this model can simulate wave propagating from shallow water to deep water ( $kh=0\sim 4$ ), and can be simplified to Nwogu equations by ignoring the high order terms (higher than  $(kh)^4$ ). Hong *et al* (2009) deduced a nonlinear mathematical model based on gravity surface wave and long wave nonlinear interaction. This model is based on non-viscosity, non-rotational fluid dynamic equations. It is a new fully dispersion and nonlinear full water equation for mild slope, water elevation and current. These governing equations contain energy coefficient reflecting energy input, friction and wave breaking loss and topography coefficient reflecting the local depth variation. This model suits wave propagating from deep water to very shallow water, with long wave field and water depth varied. Zhang *et al* (2010a) extended it into third order and third order modified model (2010b). This paper aims to deduce a fifth order theoretical model.

## THEORETICAL MODEL DERIVATION

Assuming the flow is irrotational, the continuity equation can be expressed in linear Laplace equation:

$$\nabla^2 f_T + \frac{\partial^2 f_T}{\partial z^2} = 0, -h < h < h_T \quad (1)$$

where  $\tilde{\nabla}^o(\nabla/\nabla x, \nabla/\nabla y)$  is horizontal gradient operator;  $x, y$  are horizontal directions;  $z$  directs vertically;  $h_T(x, y, t)$  is free surface elevation;  $z=0$  is still water level;  $z=-h(x, y)$  is bottom boundary.

On free surface elevation  $z=h_T$  it satisfy kinematic equation, and can be expressed as:

$$\frac{\partial h_T}{\partial t} + \nabla j_T \cdot \nabla h_T = \frac{\partial j_T}{\partial z}, z = h_T \quad (2)$$

Unsteady Bernoulli equations can be used to depict the dynamic boundary conditions on free surface.

$$g h_T + \left( \frac{\partial}{\partial t} + W \right) j_T + \frac{1}{2} [\nabla j_T \cdot \nabla j_T + j_{Tz}^2] = G(t), z = h_T \quad (3)$$

where  $g$  is gravitational acceleration;  $W$  is composed energy factor, representing energy dissipation coefficient.

Substituting Eq.(3) into (2), a new kinematic-dynamic boundary condition can be deduced:

$$\begin{aligned} g \frac{\partial j_T}{\partial z} + \left( \frac{\partial}{\partial t} + W \right) \left[ \frac{\partial j_T}{\partial t} + (\nabla j_T \cdot \nabla j_T + j_{Tz}^2) \right] \\ + \frac{1}{2} \nabla j_T \cdot \nabla (\nabla j_T \cdot \nabla j_T + j_{Tz}^2) \\ + \frac{1}{2} j_{Tz} \frac{\partial}{\partial z} (\nabla j_T \cdot \nabla j_T + j_{Tz}^2) = 0, z = h_T \end{aligned} \quad (4)$$

Since Eq.(1) is linear,  $j_T$  can be divided into wave velocity potential  $j$  and long wave velocity potential  $\varphi_c$ :

$$j_T = j(x, y, z, t) + j_c(x, y, z, t) \quad (5)$$

Here are two assumptions: 1) the vertical acceleration was omitted, i.e.  $\nabla^2 j_c / \nabla z^2 = 0$ ; 2) the vertical wave velocity was also omitted, i.e.  $\nabla j_c / \nabla z = 0$ . Adopting the above two assumptions into the equations, only the horizontal velocity  $U_c = \nabla j_c$  exist.

The wave elevation  $h_T$  can be divided into the following form:

$$h_T = h + h_c \quad (6)$$

To convert the 3 dimensional control Eq.s(1)-(5) into horizontal 2 dimensions, the 3d velocity potential function  $j(x, y, z, t)$  can be divided into local part  $F(x, y, z, t)$  and removal part  $\Phi(x, y, t)$ . The local part depicts the water local change and the removal part depicts the overall change of horizontal water area. Assuming the wave velocity potential as:

$$j(x, y, z, t) = F(x, y, z, t) \Phi(x, y, t) \quad (7)$$

Using the second order Green Function:

$$\int_{-h}^{h_T} (F j_{zz} - j F_{zz}) dz = (F j_z - j F_z) \Big|_{-h}^{h_T} = 0 \quad (8)$$

Wave can be divided from current in Eq.s (1)

$$j_{zz} = -\nabla^2 j - \nabla^2 j_c = -\nabla \cdot \nabla j - \nabla \cdot U_c \quad (9)$$

Substituting Eq. (9) into Eq. (8), we can get

$$\int_{-h}^{h_T} (F \nabla \cdot \nabla j + \nabla \cdot U_c) + j F_{zz} dz = (j F_z - F j_z) \Big|_{-h}^{h_T} = 0 \quad (10)$$

Substituting  $j F_{zz} = F j_{zz}$  into Eq. (10), we can get

$$\int_{-h}^{h_T} F (\nabla \cdot \nabla f + \nabla \cdot U_c + f_{zz}) dz = 0 \quad (11)$$

The Eq. (11) is an integration of 3 dimension continuity equation from bottom to free surface.  $F$  is weight factor.

Adopting the following form

$$F \nabla \cdot \nabla j = F \nabla \cdot (F \nabla \Phi + \Phi \nabla F) = \nabla \cdot (F^2 \nabla \Phi) + \Phi F \nabla \cdot \nabla F \quad (12)$$

and

$$\begin{aligned} \int_{-h}^{h_T} F \nabla \cdot \nabla j dz &= [\nabla \cdot \int_{-h}^{h_T} F^2 \nabla \Phi dz - F^2 \nabla \Phi \cdot \nabla Z] \Big|_{-h}^{h_T} \\ &+ \Phi \int_{-h}^{h_T} F \nabla \cdot \nabla F dz \end{aligned} \quad (13')$$

Eq. (10) can be expressed as:

$$\begin{aligned} \nabla \cdot \int_{-h}^{h_T} F^2 \nabla \Phi dz + \int_{-h}^{h_T} [F \nabla \cdot U_c + \Phi F_{zz}] dz \\ + \Phi \int_{-h}^{h_T} F \nabla \cdot \nabla F dz - F^2 \nabla \Phi \cdot \nabla Z \Big|_{-h}^{h_T} = (\Phi F_{F_z} - F f_z) \Big|_{-h}^{h_T} = 0 \end{aligned} \quad (14)$$

Adopting the kinematic boundary condition at  $z=h_T$  and  $z=-h$ , we can get :

$$\begin{aligned} -\{F^2 \nabla \Phi \cdot \nabla Z + (\Phi F_{F_z} - F j_z)\} \Big|_{-h}^{h_T} \\ = \{F \Big|_{h_T} \frac{Dh}{Dt} - \Phi F_{F_z} + \Phi F \nabla F \cdot \nabla Z\} \Big|_{-h}^{h_T} \\ + [F \Big|_{h_T} \frac{Dh_c}{Dt} + F \Big|_{-h} (\vec{U}_c \cdot \nabla h)] \end{aligned} \quad (15)$$

Substituting into (10), the following form can be obtained:

$$\begin{aligned} F \Big|_{h_T} \frac{Dh}{Dt} + \nabla \cdot [\int_{-h}^{h_T} F^2 dz] \nabla \Phi + (\nabla \cdot U_c) [\int_{-h}^{h_T} F dz - b] \\ + [\int_{-h}^{h_T} F F_{zz} dz - F F_z \Big|_{h_T} + J / g] \Phi \\ = -[F \Big|_{h_T} \frac{Dh_c}{Dt} + F \Big|_{-h} (\vec{U}_c \cdot \nabla h) + b \nabla \cdot U_c] = 0 \end{aligned} \quad (16)$$

where

$$J / g = \int_{-h}^{h_T} F \nabla \cdot \nabla F dz + F \nabla F \cdot \nabla Z \Big|_{-h}^{h_T} + F F_z \Big|_{-h} \quad (17)$$

By adopting the free surface dynamic boundary condition  $z=h_T$  and substituting the Eq. (7) into (3), the free surface boundary condition can be changed to:

$$gh + \left(\frac{D}{Dt} + \dot{W}\right) j + \frac{1}{2} [\nabla j \cdot \nabla j + j_z^2] = \alpha(t) \quad (18)$$

By adopting  $\varphi = F \Phi$ , another 2D control Equation can be achieved:

$$\begin{aligned} gh + [F] \left(\frac{D}{Dt} + \dot{W}\right) \Phi + \left[\frac{DF}{Dt}\right] \Phi + \frac{1}{2} [F^2 (\nabla \Phi \cdot \nabla \Phi) \\ + (F_z^2 + \nabla F \cdot \nabla F) \Phi^2 + \Phi \nabla \Phi \cdot \nabla F^2] = \alpha(t) \end{aligned} \quad (19)$$

or can be expressed as

$$gh + [F] \frac{D\Phi}{Dt} + \left(\frac{DF}{Dt} + \dot{W}F\right)\Phi + \frac{1}{2}[F^2(\nabla\Phi \cdot \nabla\Phi) + (F_z^2 + \nabla F \cdot \nabla F)\Phi^2 + (\nabla F^2 \cdot \nabla\Phi)\Phi] = \alpha(t), z = h_T \quad (20)$$

Eq.s(15),(19) are governing equation of shallow water wave propagation model on long waves.  $\Phi, \eta$  are nonlinear wave velocity potential and wave surface;  $\Phi_c, \eta_c$  and  $U$  are given wave velocity potential, wave surface and horizontal velocity component;  $h$  is depth under still water level,  $g$  is gravitational acceleration. To get explicit expression, the following expression was adopted

$$F(x, y, z, t) = \cosh k(h+z) / \cosh k(h+h_c), \quad (21)$$

$$-h \leq z \leq h_T = h + h_c$$

Dispersion coefficient  $k$  can be determined by Eq. (21)

$$gkg = gk \tanh kd = s^2 = s^2 \left[1 + \left(\frac{\dot{W}}{2s}\right)^2\right] \approx s^2 \quad (22)$$

where  $\sigma$  stands for natural frequency

$$s = (w - \mathbf{K} \cdot U_c) = -\frac{D}{Dt} \Psi(x, y, t) \quad (23)$$

$$w = -\frac{\partial}{\partial t} \Psi = s + \mathbf{K} \cdot U_c = -(\Phi_1 \frac{\partial \Phi_2}{\partial t} - \Phi_2 \frac{\partial \Phi_1}{\partial t}) / |\Phi|^2 \quad (24)$$

$$\mathbf{K} = \nabla \Psi = (\Phi_1 \nabla \Phi_2 - \Phi_2 \nabla \Phi_1) / |\Phi|^2 \quad (25)$$

$$\Phi = \text{Re}\Phi + i \text{Im}\Phi = \Phi_1 + i \Phi_2 = |\Phi| e^{i\Psi} \quad (26)$$

$$|\Phi|^2 = \Phi_1^2 + \Phi_2^2, \Psi = \text{arctg}(\Phi_2 / \Phi_1) \quad (27')$$

Adopting (20), Eq.s (15), (19) can be expressed as follows:

$$A_1 \frac{Dh}{Dt} + \nabla \cdot (A_3 \nabla \Phi) + A_2 \nabla U_c + A_4 \Phi = 0, Z = h_T \quad (28)$$

$$gh + B_1 \frac{D\Phi}{Dt} + \left[\left(\frac{D}{Dt} + \dot{W}\right)B_1 + B_4 \cdot \nabla\Phi\right]\Phi + \frac{1}{2}[B_2(\nabla\Phi \cdot \nabla\Phi) + B_3\Phi^2] = \alpha(t), z = h_T \quad (29)$$

where the coefficients are:

$$A_1 = F|^{h_T} = B_1 = \cosh k(d+h) / \cosh kd$$

$$\cong 1 + \frac{g}{k} (kh) + \frac{1}{2} (kh)^2 + \frac{1}{6} \frac{g}{k} (kh)^3 + \frac{1}{24} (kh)^4 + \frac{1}{120} \frac{g}{k} (kh)^5$$

$$= 1 + \frac{g}{g} h + \frac{1}{2} (kh)^2 + \frac{1}{6} \frac{g}{g} k^2 h^3 + \frac{1}{24} (kh)^4 + \frac{1}{120} \frac{g}{g} k^4 h^5$$

$$= B_1 \quad (30)$$

$$\begin{aligned}
A_2 &= \int_{-h}^{\eta r} F dz - b = \frac{\sinh k(d+h)}{k \cosh kd} - b, b = g/k \\
&\equiv h + \frac{1}{2} gkh^2 + \frac{1}{6} k^2 h^3 + \frac{1}{24} gk^3 h^4 + \frac{1}{120} k^4 h^5 \quad (31) \\
&= h + \frac{1}{2} \frac{s^2}{g} h^2 + \frac{1}{6} k^2 h^3 + \frac{1}{24} \frac{s^2}{g} k^2 h^4 + \frac{1}{120} k^4 h^5
\end{aligned}$$

$$\begin{aligned}
A_3 &= \int_{-h}^{hr} F^2 dz = [2k(d+h) + \sinh 2(d+h)] / (4k \cosh^2 kd) \\
&\equiv \frac{6Gg}{g} + h + gkh^2 + \frac{1}{3}(1+g^2)k^2 h^3 + \frac{1}{3}gk^3 h^4 + \frac{1}{15}(1+g^2)k^4 h^5 \quad (32) \\
&= \frac{6Gg}{g} + h + \frac{s^2}{g} h^2 + \frac{1}{3}[k^2 + (\frac{s^2}{g})^2]h^3 \\
&\quad + \frac{1}{3} \frac{s^2}{g} k^2 h^4 + \frac{1}{15}(k^2 + (\frac{s^2}{g})^2)k^2 h^5
\end{aligned}$$

$$A_4 = A_{41} + A_{42} + J / g \quad (33)$$

$$A_{41} = \int_{-h}^{hr} FF_{zz} dz = k^2 A_3 \quad (34)$$

$$\begin{aligned}
A_{42} &= -FF_z |^{hr} = -k \sinh 2k(d+h) / (2 \cosh^2 kd) \\
&\equiv -(kg + (1+g^2)k^2 h + 2gk^3 h^2 + \frac{2}{3}(1+g^2)k^4 h^3 \\
&\quad + \frac{2}{3}gk^5 h^4 + \frac{2}{15}(1+g^2)k^6 h^5) \quad (35) \\
&= -[\frac{s^2}{g} + [k^2 + (\frac{s^2}{g})^2]h + 2\frac{s^2}{g}(kh)^2 + \frac{2}{3}[k^2 \\
&\quad + (\frac{s^2}{g})^2]k^2 h^3 + \frac{2}{3}\frac{s^2}{g}(kh)^4 + \frac{2}{15}[k^2 + (\frac{s^2}{g})^2]k^4 h^5]
\end{aligned}$$

$$J / g = \int_{-h}^{hr} F \nabla \cdot \nabla F dz + F \nabla F \cdot \nabla Z |_{-h}^{hr} + FF_z |_{-h} \quad (36)$$

$$\begin{aligned}
B_2 &= F^2 |^{hr} = [1 + \cosh 2k(d+h)] / 2 \cos^2 kd \\
&\equiv 1 + 2gkh + (1+g^2)(kh)^2 + \frac{4}{3}g(kh)^3 \\
&\quad + \frac{1}{3}(1+g^2)(kh)^4 + \frac{4}{15}g(kh)^5 \quad (37) \\
&= 1 + 2\frac{s^2}{g}h + [k^2 + (\frac{s^2}{g})^2]h^2 + \frac{4}{3}\frac{s^2}{g}k^2 h^3 \\
&\quad + \frac{1}{3}[k^2 + (\frac{s^2}{g})^2]k^2 h^4 + \frac{2}{15}\frac{s^2}{g}k^4 h^5
\end{aligned}$$

$$B_3 = B_{31} + B_{32} \quad (38)$$

$$\begin{aligned}
B_{31} &= F_z^2 |^{hr} = k^2 [\cosh 2k(d+h) - 1] / 2 \cos^2 kd \\
&\equiv (kg)^2 + 2gk^3h + (1+g^2)k^4h^2 + \frac{4}{3}gk^5h^3 + \frac{1}{3}(1+g^2)k^6h^4 + \frac{4}{15}gk^7h^5 \\
&= \left(\frac{S^2}{g}\right)^2 + 2\frac{S^2}{g}k^2h + [k^2 + \left(\frac{S^2}{g}\right)^2](kh)^2 + \frac{4}{3}\frac{S^2}{g}k^4h^3 \\
&+ \frac{1}{3}[k^2 + \left(\frac{S^2}{g}\right)^2](kh)^4 + \frac{4}{15}\frac{S^2}{g}k^6h^5 \quad (39)
\end{aligned}$$

$$\begin{aligned}
B_{32} &= \nabla F \cdot \nabla F |^{hr} \\
&= \frac{1}{2 \cosh^2 kd} \{ [\cosh 2k(d+h) - 1] [\nabla k(d+h) \cdot \nabla k(d+h)] \\
&+ g^2 [\cosh^2 2k(d+h) - 1] (\nabla kd \cdot \nabla kd) - 2g \sinh 2k(d+h) [\nabla k(d+h) \cdot \nabla kd] \} \\
&\equiv \{g^2 + 2gkh + (1+g^2)(kh)^2 + \frac{4}{3}g(kh)^3 + \frac{(1+g^2)}{3}(kh)^4 \\
&+ \frac{4}{15}g(kh)^5\} (\nabla kh \cdot \nabla kh) + 2(1-g^2)khg + kh + \frac{2}{3}g(kh)^2 + \frac{1}{3}(kh)^3 \\
&+ \frac{2}{15}g(kh)^4 (\nabla kh \cdot \nabla kd) + (1-g^2)^2(kh)^2 \{1 + \frac{1}{3}(kh)^2\} (\nabla kd \cdot \nabla kd) \quad (40) \\
&= [(1-g^2)^2(kh)^2(\nabla kd \cdot \nabla kd) + 2g(1-g^2)kh(\nabla kd \cdot \nabla kd) + g^2(\nabla kh \cdot \nabla kh)] \\
&+ [2(1-g^2)(kh)^2(\nabla kh \cdot \nabla kd) + 2gkh(\nabla kh \cdot \nabla kh)] \\
&+ [\frac{1}{3}(1-g^2)^2(kh)^4(\nabla kd \cdot \nabla kd) + \frac{4}{3}g(1-g^2)(kh)^3(\nabla kh \cdot \nabla kd) \\
&+ (1+g^2)(kh)^2(\nabla kh \cdot \nabla kd)] + [\frac{2}{3}(1-g^2)(kh)^4(\nabla kh \cdot \nabla kd) \\
&+ \frac{4}{3}g(kh)^5(\nabla kh \cdot \nabla kh)]
\end{aligned}$$

$$\begin{aligned}
B_4 &= F \nabla F |^{hr} \\
&= \frac{1}{2 \cosh^2 kd} \{ \sinh 2k(d+h) \nabla k(d+h) - g \cosh 2k(d+h) + 1 \} \nabla kd \\
&\equiv \{g + (1+g^2)kh + 2g(kh)^2 + \frac{2}{3}(1+g^2)(kh)^3 + \frac{2}{3}g(kh)^4 \\
&+ \frac{2}{15}(1+g^2)(kh)^5\} \nabla kh + (1-g^2)kh[1 + gkh + \frac{2}{3}(kh)^2 + \frac{g}{3}(kh)^3 \\
&+ \frac{2}{15}(kh)^4] \nabla kd \\
&= [(1-g^2)kh \nabla kd + g \nabla kh] + kh [g(1-g^2)kh \nabla kd + (1+g^2) \nabla kh] \\
&+ 2(kh)^2 [\frac{1}{3}(1-g^2)kh \nabla kd + g \nabla kh] + \frac{1}{3}(kh)^3 [g(1-g^2)kh \nabla kd \\
&+ 2(1+g^2) \nabla kh] + \frac{2}{3}(kh)^4 [\frac{1}{5}(1-g^2)kh \nabla kd + g \nabla kh] \quad (42)
\end{aligned}$$

The exactness of the mathematical model is determined by the order of the coefficients. Choosing higher order of coefficients yields more accurate model results and it demands high calculation ability. Keep the third order coefficients, we can get former the third order solutions.

## SPECIAL CASE ANALYSIS

This model can be transformed into equivalent nonlinear mild slope equations for slowly varied current, water level and depth and Boussinesq-type shallow water nonlinear wave model for a relative shallow water depth and Airy shallow water Equations.

### Nonlinear deep water waves (mild-slope type)

when  $kh \gg 1$ ,  $\sigma^2/g = k$ ,  $c_g = c/2$ ,  $cc_g = g/2k$ ,  $J=0$ , the governing equations can be simplified to the mild-slope type equations :

$$(1 + kh + \frac{1}{2}k^2h^2) \frac{Dh}{Dt} + (1 + \frac{1}{2}kh + \frac{1}{6}k^2h^2)h\nabla \cdot \vec{U}_c$$

$$+ \nabla \cdot [(\frac{1}{2} + kh + k^2h^2)\nabla\Phi / k] - (\frac{1}{2} + kh + k^2h^2)k\Phi = 0 \quad (43)$$

and

$$gh + (\frac{D}{Dt} + \dot{W})(1 + kh + \frac{1}{2}k^2h^2)\Phi$$

$$+ (\frac{1}{2} + kh)(\nabla\Phi \cdot \nabla\Phi + k^2\Phi^2) = \mathcal{Q}(t) \quad (44)$$

### Shallow water nonlinear long wave equations ( Boussinesq-type )

When  $kh < 1, \sigma^2/g = k^2h[1 - (kh)^2/3], cc_g = gh[1 - 2(kd)^2/3], k^2cc_g - \sigma^2 = -gk^4h^3/3, \sigma^4/g = gk^4h^2[1 - 2(kh)^2/3]$  on slowly varied current, water level and water depth, approximately take  $\tilde{N} \cdot \tilde{N}\Phi = -k^2\Phi$ , the governing equations could be transformed to Boussinesq-type nonlinear shallow water long wave equations:

$$\left[ 1 - (h + \frac{h}{2})(\nabla \cdot \nabla h) - \frac{h^3}{3} \nabla \cdot \nabla(\nabla \cdot \nabla h) \right] \frac{\partial h}{\partial t} +$$

$$\nabla \cdot \left[ (h + h)\nabla\Phi + \left( \frac{h^3}{3} - hh^2 - \frac{h^3h(\nabla \cdot \nabla h)}{3} \right) \nabla(\nabla \cdot \nabla\Phi) \right] + \quad (45)$$

$$\frac{J\Phi}{g} - \left[ (h + h)hh + \frac{1}{3}h^3(2h + h)\nabla \cdot \nabla h \right] \nabla \cdot \nabla\nabla \cdot \nabla\Phi +$$

$$\nabla(\nabla \cdot \nabla\Phi) \cdot \nabla\left(\frac{h^3}{3}\right) = \alpha(em^4, e^2m^2)$$

and

$$gh + (\frac{\partial}{\partial t} + \dot{W}) \left[ \Phi - (hh + \frac{h^2}{2} + \frac{h^3}{3} \nabla \cdot \nabla h)(\nabla \cdot \nabla\Phi) \right] +$$

$$\left[ \frac{1}{2} - h\nabla \cdot \nabla h - \frac{h^3}{3} \nabla \cdot \nabla(\nabla \cdot \nabla h) \right] (\nabla\Phi \cdot \nabla\Phi) + \quad (46)$$

$$\left( \frac{h^2}{2} + hh \right) (\nabla \cdot \nabla\Phi)^2 + \left( \frac{h + h}{3} \right) h^3 \nabla(\nabla \cdot \nabla\Phi) \cdot \nabla(\nabla \cdot \nabla\Phi)$$

$$= \alpha(em^4, e^2m^2)$$

### Nonlinear long wave equations for very shallow water ( Airy wave )

when  $kh \ll 1, \sigma^2/g = k^2h, c_g = c = (gh)^{1/2}, k^2cc_g = g^2kh$ , the governing equations could be simplified to the Airy wave equations:

$$[1 + (h + \frac{1}{2}h)k^2h] \frac{Dh}{Dt} + (h + \frac{1}{2}k^2hh^2 + \frac{1}{6}k^2h^2)\nabla \cdot \vec{U}_c$$

$$+ \nabla \cdot [(h + h + hk^2h^2)\nabla\Phi] - [\frac{J}{g} - k^4hk(h + h)]\Phi = 0 \quad (47)$$

and

$$gh + (\frac{D}{Dt} + \dot{W}) [\Phi + (h + \frac{h}{2})hk^2\Phi]$$

$$+ (\frac{1}{2} + hk^2h)\nabla\Phi \cdot \nabla\Phi + (\frac{h^2}{2} + hh)k^4\Phi^2 = \mathcal{Q}(t) \quad (48)$$

Omitted the high order terms containing  $J$  and  $kh$  of the above two equations, we can obtain the nonlinear long wave equations for very shallow water (i.e. Airy shallow water long wave equations) as follows:

$$\left(\frac{D}{Dt} + \nabla \cdot \mathbf{U}_c\right)h + \nabla \cdot [(h + h)\nabla\Phi] = 0 \quad (49)$$

$$gh + \left(\frac{D}{Dt} + \dot{W}\right)\Phi + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi = \alpha(t) \quad (50)$$

## CONCLUSIONS

- 1) A fully dispersive fifth order nonlinear wave propagation model for mild current, water level and depth was established theoretically considering energy and topography factors.
- 2) By omitting the 5th order terms; the lower order model was in consistency with the former 3rd order model.
- 3) This model could be simplified to mild-slope type equations for deep water, to Boussinesq type equations for shallow water and to Airy wave for very shallow water.
- 4) The efficiency and accuracy of the mathematical model were yet to be validated.

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