Linear Integral Boundary Value Problem of a Class of Evolution Inclusions

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Abstract. In this paper, we establish the existence of solutions of a class of evolution inclusions with a multivalued perturbation under linear integral boundary value conditions. By applying the Leray -Schauder's theorem we can investigate the existence of solutions for the nonconvex cases and convex cases on the multivalued perturbation.

Introduction

In recent years, the solvability problem for integral boundary value conditions of evolution equation has been one of the important research field at home and abroad, and a number of outstanding research results have been produced, see [1-4, 7-10]. However, relevant results of the problem of solvability for integral boundary value conditions of evolution inclusions are very few. Therefore, this paper studies the solvability for integral boundary value conditions of the evolution inclusions: $y'(t)+By\in F(t,\ y)$, where F is a multifunction and B is a bounded linear operator.

Preliminaries

For convenience, we present some notations and basic concepts as follows. In Euclidean space R^N, $\langle\cdot,\cdot\rangle$ expresses an inner product, and $|\ \cdot\ |$ expresses the Euclidean norm. Let $L=[0,2\pi]$, we denote the set of the map $y:L\to R^N$ which satisfies $\int_0^{2\pi}|y|^2dt<\infty$ in $K^2(L,R^N)$, and the norm is denoted by $\|y\|_2=\left(\int_0^{2\pi}|y|^2dt\right)^{\frac{1}{2}}$.

Let

$$C_{2\pi}=\{\ y|L\mapsto R^N \text{ is continuous; } y(0)=\alpha\int_0^{2\pi}y(s)ds,\ 0<\alpha<1\ \}, \tag{1}$$

$$V^{1,2}(L,R^N)=\left\{\ y\in C_{2\pi}:\int_0^{2\pi}\left(|y|^2+|y'(t)|^2\right)dt<\infty\right\}, \tag{2}$$

where y' is the derivative of y. $C_{2\pi}$ and $V^{1,2}(L,R^N)$ are Banach space , which the norm are

$$\|\ y\ \|_c=\max_{t\in L}|\ y\ | \text{ and } \|\ y\ \|_{1,2}=\left(\int_0^{2\pi}\left(|y|^2+|y'(t)|^2\right)dt\right)^{\frac{1}{2}}. \tag{3}$$

Let $[L,\Sigma]$ be the Lebesgue measurable space and Y be a separable Banach space.

Let $M \subset Y$ and Y is closed, $y \in Y$, then the distance form y to M is defined as

$$d(y,M) = \inf\left\{ \left|y-m\right| : m \in M \right\}. \tag{4}$$

We define $P_f(Y) = \{ M \subset Y$:nonempty and closed}, a multifunction $F: L \to P_f(Y)$ is measurable if and only if, for every $z \in Y$, the function $t \to d(z, F(t)) = \inf\{ \|z-y\| : y \in F(t) \}$ is measurable. A multifunction $G: L \to 2^Y \setminus \{\phi\}$ is said to be graph measurable if $GrG = \{(t,y) : y \in G(t)\} \in \Sigma \times N(Y)$ with $N(Y)$ being the Borel $\sigma-$field of Y. On $P_f(Y)$ we define a generalized metric as the'Hausdorff metric', by setting

$$h(M,N) = \max\left\{ \sup_{m\in M} d(m,N), \sup_{n\in N} d(n,M) \right\}, \tag{5}$$

for all $M, N \in P_f(Y)$. As we all known, $(P_f(Y), h)$ is a complete metric space.

Let X, Z be Hausdorff topological spaces and $G: X \to 2^z \setminus \{\phi\}$. $G(\cdot)$ is 'upper semicontinuous' (resp. 'lower semicontinuous'), if for all $C \subseteq Z$ nonempty and closed, $G^-(C) = \{ x \in X : G(x) \cap C \neq \phi \}$ (resp. $(G^+(C) = \{x \in X : G(x) \subseteq C\})$) is closed in X.

Lemma 2.1. Consider the equation

$$\begin{cases} y'(t) + By = f(t), \\ y(0) = \alpha \int_0^{2\pi} y(s)ds, \end{cases} \tag{6}$$

Where B: $R^N \to R^N$ is a bounded linear operator, $0 < \alpha < 1$, and there exists $c \in R^+$ such that $\langle By, y \rangle \geq c|\ y\ |^2$ where $c > \dfrac{1}{\alpha}$ for all $y \in R^N$, then the problem (6) has a unique solution and satisfies $\|y\|_2 \leq C\|f\|$, where C is a constant.

Proof: First, we consider the following Cauchy problem

$$\begin{cases} y'(t) + By = f(t), \\ y(0) = \lambda, \end{cases} \tag{7}$$

Where $\lambda \in R^N$. It is easy to know that the problem (7) has a unique solution written as follows:

$$y(t) = e^{Bt}\lambda + \int_0^t e^{B(t-s)} f(s)ds. \tag{8}$$

Integrating the equation (8) from 0 to 2π, we have that

$$\int_0^{2\pi} y(t)dt = \int_0^{2\pi} e^{Bt}\lambda dt + \int_0^{2\pi}\int_0^t e^{B(t-s)} f(s)dsdt. \tag{9}$$

Let $y(0) = \alpha\int_0^{2\pi} y(t)dt$, so $\lambda = \alpha\int_0^{2\pi} y(t)dt$, by (9), we have that

$$\frac{1}{\alpha}\lambda = \int_0^{2\pi} e^{Bt}dt\lambda + \int_0^{2\pi}\int_0^t e^{B(t-s)} f(s)dsdt. \tag{10}$$

And by $\langle By, y\rangle \geq c| \ y \ |^2$, then there exists $\left(\frac{1}{\alpha}-\int_0^{2\pi} e^{Bt}dt\right)^{-1}$, thus we obtain that

$$\lambda=\left(\frac{1}{\alpha}-\int_0^{2\pi} e^{Bt}dt\right)^{-1}\times\int_0^{2\pi}\int_0^t e^{B(t-s)}f(s)dsdt \quad , \tag{11}$$

So we get the following unique solution (12) by taking (11) into (8)

$$y(t)=e^{Bt}\left(\frac{1}{\alpha}-\int_0^{2\pi} e^{Bt}dt\right)\times\int_0^{2\pi}\int_0^t e^{B(t-s)}f(s)dsdt+\int_0^t e^{B(t-s)}f(s)ds \quad . \tag{12}$$

Taking the norm on both sides of (12), we have

$$|y(t)|\leq\left|e^{Bt}\left(\frac{1}{\alpha}-\int_0^{2\pi} e^{Bt}dt\right)\right|\cdot 2\pi\left|\int_0^{2\pi} e^{B(t-s)}f(s)ds\right|+\left|\int_0^{2\pi} e^{B(t-s)}f(s)ds\right|$$

$$\leq\left(M_1 2\pi+1\right)\int_0^{2\pi}\left|e^{B(t-s)}f(s)\right|ds$$

$$=M_2\int_0^{2\pi}\ |f(s)|ds \ ,$$

Where M_1 and M_2 are constants. Therefore, there exists a constant $C>0$ for $\| \ y \ \|_2 \leq C\| \ f \ \|_2$. This completes the proof of lemma 2.1.

Main Results

Consider the following evolution inclusion problem:

$$\begin{cases} y'(t)+By\in F(t,y), \\ y(0)=\alpha\int_0^{2\pi} y(s)ds, \end{cases} \tag{13}$$

Where $B: R^N \to R^N$ is a bounded linear operator, $0<\alpha<1$,and there exists $c\in R^+$ such that $\langle By, y\rangle \geq c| \ y \ |^2$ where $c>\frac{1}{\alpha}$ and $F: L\times R^N \to 2^{R^N}$ is a multifunction.

Definition 3.1. y Is called the solution of the problem (13), if a function $y\in V^{1,2}(L,R^N)$ and there exists a function $f(t)\in F(t,y(t))$ such that

$$\langle y'(t),w\rangle+\langle By,w\rangle=\langle f(t),w\rangle \tag{14}$$

For all $w\in R^N$ and almost $t\in L$.

Next, we prove the existence of solutions of the problem (13) under the following hypothesis G_1 when the multivalued F is nonconvex-valued.

G_1: $F: L\times R^N \to R^N$ is a multifunction with compact such that

(i) $(t,y)\to F(t,y)$ is graph measurable;

(ii) For almost all $t\in L, y\to F(t,y)$ is lower semicontinuous;

(iii) There exists an nonnegetive function $n(\cdot)\in K_{+}^{2}(L)$ and a constant $c_1>0$ such that $|F(t,y)|=\sup\{\ \|f\|: f\in F(t,y)\}\le n(t)+c_1|y|^{\beta}$, for all $y\in R^N$, where $\beta<1$.

Theorem 3.1. If hypothesis G_1 hold, then the problem (13) has a solution $y\in V^{1,2}(L,R^N)$.

Proof Let $Ky=y'+By$, for all $y\in V^{1,2}(L,R^N)$, then $K:V^{1,2}(L,R^N)\to K^2(L,R^N)$ is a linear operator. By Lemma 2.1, one priori estimate to the solution of equation and $V^{1,2}(L,R^N)$ embeds compactly into $K^2(L,R^N)$, $K:V^{1,2}(L,R^N)$ then $\to K^2(L,R^N)$ is one to one, and because K is continuous, so we have $K^{-1}:K^2(L,R^N)\to V^{1,2}(L,R^N)$ is a continuous operator. Thanks to $V^{1,2}(L,R^N)$ embedding compactly into $K^2(L,R^N)$, so $K^{-1}:K^2(L,R^N)\to K^2(L,R^N)$ is completely continuous.

Next, let $Q:K^2(L,R^N)\to 2^{K^2(L,R^N)}$ be the multivalued Nemitsky operator corresponding to be defined by $Q(y)=\{\ w\in K^2(L,R^N): w(t)\in F(t,y(t))\}$. By Theorem 3.1 of [5], $Q(\cdot)$ has nonempty, closed, decomposable values and is lower semicontinuous.

By continuous selection theorem of [6] and obtain a continuous map $f:K^2(L,R^N)\to\ K^2(L,R^N)$ such that $f(y)\in F(t,y)$. To finish our proof, we only need to solve the fixed point problem: $y=K^{-1}\circ f(y)$.We apply Leray-Schauder's alternative theorem to prove the fixed point problem. It is equivalent to test the set $W=\{\ y\in K^2(L,R^N): y=\sigma K^{-1}f(y),\sigma\in(0,1)\}$ is bounded. let $y\in W$, then $y=\sigma K^{-1}\circ f(y)$. By hypothesis $G_1(iii)$, we can derive $|f(y)|\le n(t)+c_1|y|^{\beta}$,then

$$\|f(y)\|_2\le\|n\|_2+\left\|c_1|y|^{\beta}\right\|_2$$

$$\le\|n\|_2+\left(\left\|c_1{}^2\right\|_{\frac{1}{1-\beta}}\left\||y|^{2\beta}\right\|_{\frac{1}{\beta}}\right)^{\frac{1}{2}}$$

$$=\|n\|_2+\|c_1\|_{\frac{2}{1-\beta}}\|y\|_2^{\beta}$$

$$=\|n\|_2+c_1(2\pi)^{\frac{1-\beta}{2}}\|y\|_2^{\beta}.$$

By Lemma 2.1, we get that

$$\|y\|_2\le c_2\|f\|_2,$$

for some constant $c_2>0$. So, we have that

$$\|y\|_2\le c_2\|n\|_2+c_1c_2(2\pi)^{\frac{2}{1-\beta}}\|y\|_2^{\beta}. \tag{15}$$

Due to $0<\beta<1$, there exists a constant $M>0$ such that $\|y\|_2\le M$. If the constant M does not exist, by (15), we have

$$1 \le \frac{c_2\|n\|_2}{\|y\|_2} + \frac{c_1 c_2 (2\pi)^{\frac{2}{1-\beta}}}{\|y\|_2^{1-\beta}} . \tag{16}$$

When $\|y\|_2$ is sufficiently large, the right end of (16) is smaller than 1. Therefore, the assumption does not established. So the set W is bounded in $K^2(L,R^N)$. Due to B is a bounded linear operator, thus, there exists a constant $c_3 > 0$ such that $\| y \|_{1,2} \le c_3$. According to Leray-Schauder's alternative theorem, there exists a $y \in V^{1,2}(L,R^N)$ such that $y = K^{-1} \circ f(y)$, then y is a solution of the problem (13).

G_2: $F: L \times R^N \to R^N$ is a multifunction with compact and convex value such that

(i) $(t,y) \to F(t,y)$ is graph measurable;

(ii) For almost all $t \in L$, $y \to F(t,y)$ has a closed graph ;

(iii) There exists an nonnegetive function $n(\cdot) \in K_+^2(L)$ and a constant $c_1 > 0$ such that $|F(t,y)| = \sup\{ \|f\| : f \in F(t,y)\} \le n(t) + c_1 |y|^{\beta}$, for all $y \in R^N$, where $\beta < 1$.

Theorem 3.2. If hypothesis G_2 hold, then the problem (13) has a solution $y \in V^{1,2}(L,R^N)$,and the solution set is weakly compact in $V^{1,2}(L,R^N)$.

Proof : By Theorem 3.1, $K^{-1}: K^2(L,R^N) \to K^2(L,R^N)$ is completely continuous, multivalued Nemitsky operator $Q: K^2(L,R^N) \to 2^{K^2(L,R^N)}$ is nonempty, closed and convex .

Next, we prove that $Q(\cdot)$ is upper semicontinuous from $V^{1,2}(L,R^N)$ to $K^2(L,R^N)_w$. Let Q_α be a nonempty and weakly closed subject of $K^2(L,R^N)$. We need to show that set $Q^{-1}(Q_\alpha) = \{ y \in E(K) : Q(y) \cap Q_\alpha \ne \phi \}$ is closed. We assume that there is function sequence $\{y_n\}_{n\ge1} \subseteq Q^{-1}(Q_\alpha)$ satisfies $y_n \to y$ in $V^{1,2}(L,R^N)$. So there is a convergent subsequence, we can get $y_n(t) \to y(t)$. Let $f_n \in Q(y_n) \cap Q_\alpha$, $n \ge 1$. Due to hypothesis $G_2(iii)$ and the Dunford-Pettis theorem, then $f_n \to f$ weakly in $K^2(L,R^N)$. As before, we have

$$f(t) \in \overline{conv}\,\overline{\lim}\{f_n(t)\}_{n\ge1} \subseteq \overline{conv}\,\overline{\lim} F(t,y) \subseteq F(t,y), \tag{17}$$

For almost all $t \in L$, where $\overline{conv}$ expresses closed convex inclusion of set, then $f \in Q(y) \cap Q_\alpha$, so $Q^{-}(Q_\alpha)$ is closed in $V^{1,2}(L,R^N)$. Therefore, $Q(\cdot)$ is upper semicontinuous from $V^{1,2}(L,R^N)$ into $K^2(L,R^N)_w$. This ends the proof.

The problem (13) is equivalent to following fixed point problem:

$$y \in K^{-1} \circ Q(y) .$$

Recalling that $K^{-1}: K^2(L,R^N) \to K^2(L,R^N)$ is completely continuous, $K^{-1} \circ Q: K^2(L,R^N) \to K^2(L,R^N)$ is USC and maps bounded sets into relatively compact sets. as in the proof of Theorem 3.1. We easily check that the set

$$\Gamma_1 = \{ y \in K^{-1}(L,R^N) : y \in \sigma K^{-1} Q(y), \sigma \in (0,1) \} \tag{18}$$

Is bounded, Invoking set value Leray-Schauder's alternative theorem, there exists $y \in V^{1,2}(L, R^N)$ such that $y \in K^{-1} \circ Q(y)$. Evidently, y is a solution of the problem (13).

Let S_P is the solution set of the problem (13). As in the proof of Theorem 3.1, we get a priori estimate to the solution. Then we have that $|S_P| = \sup\left\{ \|u\|_{1,2} : u \in S_P \right\} \le M$, where $M > 0$. By hypothesis $G_2(iii)$ and Dunford-Pettis theorem, we may assume that $y_n \to y$ in $V^{1,2}(L, R^N)$. As before, for almost all $t \in L$, we have

$$Ky \in \overline{conv}\,\overline{\lim}\{Ky_n\}_{n\ge 1} \subseteq \overline{conv}\,\overline{\lim} F(t, y_n) \subseteq F(t, y) \tag{19}$$

,

Then $y \in S_P$, hence S_P is weakly compact in $V^{1,2}(L, R^N)$. The proof is completed.

References

[1] H.Hua, F.Cong and Y.Cheng: Applied Mathematics Letters, Vol. 25 (2012) No.12, p.2423-2428.

[2] X.L.Liu and X.Y.Shi: Applied Mathematics Letters, Vol. 37 (2014),p.91-94.

[3] Y.S.Xu , J.R.Mu and X.L.Liu : Far East Journal of Applied Mathematics, Vol. 92 (2015) No.2,p. 153.

[4] A.Cabada and G.Wang: Journal of Mathematical Analysis and Applications, Vol. 389 (2012)No. 1, p.403-411.

[5] Y.Cheng , F.Cong and H.Hua: Advances in Difference Equations, Vol. 165 (2012),p.1-12.

[6] J.Dugundji and A.Granas: Monogr Matematyczne, Vol. 123 (1986),p.9-31.

[7] Y.Wang and Y.M.Shi: Math Anal Appl, Vol. 309 (2005),p.56-69.

[8] A.Belarbi , M.Benchohra and A.Ouahab: Multiple positive solutions for nonlinear boundary value problems with integral boundary conditions. Archivum Mathematicum, Vol. 44(1)(2008), p.1-7.

[9] M.Benchohra, S.Hamani and JJ.Nieto: The method of upper and lower solutions for second order differential inclusions with integral boundary conditions. The Rocky Mountain Journal of Mathematics, Vol. 40(1)(2010), p.13-26.

[10] B.Ahmad, JJ.Nieto: Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions. Boundary Value Problems, Vol. 2009(2009),p.1-11.

