

# Stability of CDFEs With Multiple Known and Unknown Delays: Discretized LKF Approach

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**Abstract**—This article considers the stability problem of a class of coupled differential-functional equations (CDFEs) with multiple delays via the discretized Lyapunov-Krasovskii functional (LKF) approach. The system discussed contains both distributed delays known and discrete delays unknown, a LKF with two parts is chosen, where one part is a complete quadratic LKF which involves those known delays only and can be done discretization, the other is a simple LKF for the unknown delays. Through independently divided every known delay region that the plane regions consists in two known delays to discretized the LKF, the stability conditions of the systems are established based on a linear matrix inequality (LMI), which is dependent with the known delays, is independent with the unknown. Finally, Two examples are given to illustrate that the result is reliable and effective.

**Keywords**—discretized LKF approach; coupled differential-functional equations; multiple unknown discrete delays; multiple known distributed delays

## I. INTRODUCTION

The coupled differential-functional equations

$$\begin{aligned}\dot{x}(t) &= f(t, x(t), y_t), \\ y(t) &= g(t, x(t), y_t)\end{aligned}$$

offers significant advantage over the traditional model of retarded and neutral type. Typically, the number of state variables in the practical systems may be rather large, but only a few components involve delay elements and are often low dimensional, i.e.  $\dim(x(t)) \gg \dim(y(t))$ . It is well known that many electrical, fluid and other systems initially described by partial differential equations can be arrived at a set of coupled differential-difference equations (CDDEs) as lossless propagation model [1][2], steam pressure control problem and water transmission problem [3]. Up to early 1990s, such systems were analyzed by taking the systems in a class of standard DDEs of neutral type. A nice overview is to analysis it directly as [4]. Related to most important works can be found in the references [5][6][7]. Gu and Niculescu [5] showed the existence of a quadratic LKF is necessary and sufficient for the stability of linear CDDE with single delay. Gu and Liu [9] generalized the difference equations to functional equations under the assumption of input-to-output stability (IOS) of the equations, and considered the uniform asymptotic stability of the systems. [6] also formulated a discretized LKF method based on the

quadratic functional in [5]. Gu [7] extend the formulation of single delay to the case of multiple delay channels, and [10] generalized the analytical results of [5][6].

A special form of CDFEs

$$\begin{aligned}\dot{x}(t) &= f_0(t, x(t), y_t) + \int_{-r}^0 H(\sigma)y(t+\sigma)d\sigma, \\ y(t) &= g_0(t, x(t), y_t) + \int_{-r}^0 M(\sigma)y(t+\sigma)d\sigma\end{aligned}$$

are more general than CDDEs. Specially,  $H(\sigma)$  and  $M(\sigma)$  in  $[-r, 0)$  may be bounded, piecewise constant matrices. Such a system may be written as a forward system, and

$$u_i(t) = K_i \int_{-r_i}^0 y(t+\theta)d\theta, i = 1, 2, \dots, N$$

as the feedback. Closely, Li [11] discussed the stability of CDFEs with a discrete and distributed delays via discretized LKF approach. In engineering practice, however, some of the time-delays for the systems are known or can be measured in a finite region, the others are difficult to measure, even if they can be estimated, is also very big. So we hope to control known delays in input to stabilizing system. In this paper we will investigate the stability of such CDFEs with multiple known and unknown discrete delays and known distributed delays. And we wish to set up the stability condition with a well robustness, which is dependent with the known or small delays, and is independent with those unknown or large delays.

In this article,  $\mathbb{R}^n$  and  $\mathbb{R}^{m \times n}$  represent the set of real  $n$ -vectors and  $m$  by  $n$  matrices, respectively.  $\mathbb{R}_+$  denote the sets of positive real numbers. For a given delays  $r_i \in \mathbb{R}_+ (i = 1, 2, \dots, N)$ ,  $\mathcal{PC}([-r, 0), \mathbb{R}^m)$  is the space of bounded, right continuous, and piecewise continuous functions defined on  $[-r, 0) (r = \max_{1 \leq i \leq N} \{r_i\})$ . These spaces are equipped with the  $L_2$  norm  $\|\phi\|_{L_2}^2 = \int_{-r}^0 \|\phi(\theta)\|^2 d\theta$ . For a given function  $y(t) \in \mathbb{R}^m$  and  $\sigma \in \mathbb{R}$ , we define  $y_\sigma \in \mathcal{PC}$  by  $y_\sigma(\theta) = y(\sigma+\theta)$ ,  $\theta \in [-r, 0)$ .  $\|\cdot\|$  denotes 2-norm for vectors and matrices.  $\lambda_{\min}(M)$  denotes the minimum in eigenvalues of Matrix  $M$ .  $(R_{ij})_{i=i_1 \rightarrow i_m}^{j=j_1 \rightarrow j_n}$  denotes a block matrix with some block matrices  $R_{i_k j_l}$ , where the line index  $i$  is from  $i_1$  to  $i_m$  and the column index  $j$  is from  $j_1$  to  $j_n$ .

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II. PROBLEM STATEMENT

In this section, we discuss the system described by the following coupled functional-differential equations

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N+M} B_i y(t-r_i) + \sum_{i=1}^N H_i \int_{-r_i}^0 y(t+\theta) d\theta, \quad (1)$$

$$y(t) = Cx(t) + \sum_{i=1}^{N+M} D_i y(t-r_i) + \sum_{i=1}^N M_i \int_{-r_i}^0 y(t+\theta) d\theta \quad (2)$$

with the initial condition

$$x(t_0) = \psi, \quad y_t = \phi.$$

where  $x \in \mathbb{R}^n, y \in \mathbb{R}^m$  are state vectors,  $r_i \in \mathbb{R}_+$  ( $i = 1, 2, \dots, N$ ) is some known delays,  $r_{N+j} \in \mathbb{R}_+$  ( $j = 1, 2, \dots, M$ ) is some unknown delays and  $r = \max_{1 \leq i \leq N+M} \{r_i\}$ ,  $A \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{m \times n}, B_i, H_i \in \mathbb{R}^{n \times m}, D_i, M_i \in \mathbb{R}^{m \times m}, i = 1, 2, \dots, K$ , are some known real matrices.

The Lyapunov-Krasovskii functional method is an effective method of studying the stability of a system described by differential-functional equations. Choose the LKF as

$$V(x(t), y_t) = V_1 + V_2, \quad (3)$$

where

$$\begin{aligned} V_1 &= x^T(t)Px(t) + 2x^T(t) \sum_{i=1}^N \int_{-r_i}^0 Q_i(\eta)y(t+\eta)d\eta \\ &+ \sum_{i=1}^N \sum_{j=1}^N \int_{-r_i}^0 \int_{-r_j}^0 y^T(t+\xi)R_{ij}(\xi,\eta)y(t+\eta)d\eta d\xi \\ &+ \sum_{i=1}^N \int_{-r_i}^0 y^T(t+\xi)S_i(\xi)y(t+\xi)d\xi, \\ V_2 &= \sum_{j=N+1}^{N+M} \int_{-r_j}^0 y^T(t+\xi)\bar{S}_j y(t+\xi)d\xi. \end{aligned}$$

*Remark 1:* In the LKF (3),  $V_1$  is a completely quadratic LKF which may be manipulate to discretized and respect only to the known delays,  $V_2$  is a simple LKF and relate to those unknown or large delays.

For the LKF of (3), it is no difficult to induce the following proposition by Theorem 3 in [6].

*Proposition 2:* Suppose the difference-integral equation (DIE) of (2) is input-to-output stable. If there exist symmetric positive definite matrix  $P$ , and continuous matrix functions  $Q_i(\eta), S_i(\eta)$  on  $[-r_i, 0)$  and  $R_{ij}(\xi, \eta) = R_{ji}^T(\xi, \eta)$  on  $[-r_i, 0) \times [-r_j, 0)$  ( $i, j=1, 2, \dots, N$ ) and  $\bar{S}_i$  on  $[-r_{N+k}, 0)$  ( $k = 1, 2, \dots, M$ ) such that the LKF of (3) satisfy

$$V(x(t), y_t) \geq \varepsilon \|x(t)\|^2, \quad (4)$$

$$\dot{V}(x(t), y_t) \leq -\varepsilon \|x(t)\|^2 \quad (5)$$

for some  $\varepsilon > 0$ , then the systems described by (1)-(2) is asymptotically stable.

III. STABILITY CONDITION OF DIE

In Proposition 2, IOS of (2) is a necessary condition for the stability of the whole systems (1)-(2). Moreover, It is well known that IOS of (2) is equivalent to the stability of DIE presented by

$$y(t) = \sum_{i=1}^{N+M} D_i y(t-r_i) + \sum_{i=1}^N M_i \int_{-r_i}^0 y(t+\theta) d\theta. \quad (6)$$

If  $M_i = 0 (i = 1, 2, \dots, N)$  [9] gave a sufficient condition for stability of the difference equations described by

$$y(t) = \sum_{i=1}^{N+M} D_i y(t-r_i). \quad (7)$$

Namely, the difference equation (7) is delay-independent exponentially stable, if there exist symmetric positive definite matrices  $S_k, k = 1, 2, \dots, N + M$  such that

$$\text{diag} ( S_1 \quad S_2 \quad \dots \quad S_{N+M} ) - D^T \left( \sum_{k=1}^{N+M} S_k \right) D > 0, \quad (8)$$

where  $D = ( D_1 \quad D_2 \quad \dots \quad D_{N+M} )$ .

For given any initial function  $\phi \in \mathcal{PC}$ , we denotes the unique solution of (6) in  $y(t, \phi)$  and  $y_t(\phi) = \{y(t+\theta, \phi) | \theta \in [-r, 0)\}$  stands for the partial trajectory of the system (6). For the exponential stability of (6), we introduce the Lyapunov-Krasovskii conditions from [13].

*Proposition 3:* Supposed the difference equation (7) is delay-independent exponentially stable. If there a functional  $v : \mathcal{PC} \rightarrow \mathbb{R}$  such that  $t \rightarrow v(y_t(\phi))$  is differentiable and the following conditions hold:

$$\alpha_1 \|\phi\|_{L_2}^2 \leq v(\phi) \leq \alpha_2 \|\phi\|_{L_2}^2,$$

$$\dot{v}(y_t(\phi)) \leq -\beta \|y_t(\phi)\|_{L_2}^2,$$

for some  $\alpha_2 > \alpha_1 > 0$  and  $\beta > 0$ , then the difference-integral equation (6) is exponentially stable.

By Proposition 3, we will give a stability condition of (6) based on LMI in the following lemma.

*Lemma 4:* DIE (6) is exponentially stable if there exist positive definite matrices  $X_i \in \mathbb{R}^m, i = 1, 2, \dots, N + M$ , and  $Y_j \in \mathbb{R}^m, j = 1, 2, \dots, N$ , such that LMI

$$\begin{pmatrix} X & 0 & D^T \mathcal{G}(X, Y) \\ 0 & Y & M^T \mathcal{G}(X, Y) \\ \mathcal{G}(X, Y) D & \mathcal{G}(X, Y) M & \mathcal{G}(X, Y) \end{pmatrix} > 0 \quad (9)$$

is satisfied, where

$$X = \text{diag} ( X_1 \quad X_2 \quad \dots \quad X_{N+M} ),$$

$$Y = \text{diag} \left( \frac{Y_1}{r_1} \quad \frac{Y_2}{r_2} \quad \dots \quad \frac{Y_N}{r_N} \right),$$

$$\mathcal{G}(X, Y) = \sum_{i=1}^{N+M} X_i + \sum_{j=1}^N r_j Y_j,$$

$$D = ( D_1 \quad D_2 \quad \dots \quad D_{N+M} ),$$

$$M = ( M_1 \quad M_2 \quad \dots \quad M_N ).$$

*Proof:* In order to save space, we just write three steps of the proof process. (i)Apply Schur's compensation to (9). (ii) Choose Lyapunov functional as

$$v(t, y_t) = \sum_{i=1}^{N+M} \int_{-r_i}^0 y^T(t+\theta) X_i y(t+\theta) d\theta + \sum_{j=1}^N \int_{-r_j}^0 d\theta \int_{t+\theta}^t y^T(\tau) Y_j y(\tau) d\tau.$$

(iii)The above Lyapunov functional satisfies the first condition of Proposition 3. Use the Jensen inequality in the derivative of  $v$  to obtain the second condition. ■

*Remark 5:* Based on the characteristic function associated to (6), [13] gave a exponentially stability condition

$$\sum_{i=1}^{N+M} \|D_i\| + r \max_{1 \leq i \leq N} \|M_i\| < 1.$$

The example 2 in [13] showed that although above inequality could be easily verify, it may be pretty conservative. The example also can illustrated that (8) is less conservative than the condition  $\sum_{i=1}^{N+M} \|D_i\| < 1$  for (7).

#### IV. STABILITY CONDITIONS OF THE SYSTEMS

##### A. Derivative of LKF

In this section, we consider the stability problem of the systems (1)-(2) with the initial condition. In order to do a simple denotation in the following, define a real number sequence

$$\mathcal{X}(i) = \begin{cases} 1, & i = 1, 2, \dots, N; \\ 0, & i = N + 1, N + 2, \dots, N + M. \end{cases}$$

and a matrix sequence

$$\bar{S}(i) = \begin{cases} S_i(-r_i), & i = 1, 2, \dots, N; \\ S_i, & i = N + 1, N + 2, \dots, N + M. \end{cases}$$

Taking derivative to  $V$  along the system trajectory, we have

$$\begin{aligned} \dot{V}(x(t), y_t) = & - \sum_{i=0}^{N+M} \sum_{j=0}^{N+M} z_i^T(t) \bar{\Delta}_{ij} z_j(t) \\ & + 2 \sum_{i=1}^N x^T(t) \int_{-r_i}^0 [\Pi_{0i}(\eta) - \dot{Q}_i(\eta)] y(t+\eta) d\eta \\ & + 2 \sum_{i=1}^N \sum_{j=1}^{N+M} z_j^T(t) \int_{-r_i}^0 \Pi_{ji}(\eta) y(t+\eta) d\eta \\ & + \sum_{i=1}^N \sum_{j=1}^N \int_{-r_i}^0 \int_{-r_j}^0 y^T(t+\xi) \Lambda_{ij}(\xi, \eta) y(t+\eta) d\eta d\xi \\ & - \sum_{i=1}^N \int_{-r_i}^0 y^T(t+\eta) \dot{S}_i(\eta) y(t+\eta) d\eta \end{aligned} \quad (10)$$

where

$$z_i(t) = \begin{cases} x(t), & i = 0; \\ y(t-r_i), & 1 \leq i \leq N+M; \end{cases}$$

$$\begin{aligned} \bar{\Delta}_{00} &= -A^T P - PA - C^T \bar{S}_\Sigma(0) C \\ &\quad - \bar{Q}_\Sigma(0) C - C^T \bar{Q}_\Sigma^T(0), \\ \bar{\Delta}_{0i} &= -PB_i - \bar{Q}_\Sigma(0) D_i \\ &\quad - C^T \bar{S}_\Sigma(0) D_i + \mathcal{X}(i) Q_i(-r_i), \\ \bar{\Delta}_{ii} &= \bar{S}(i) - D_i^T \bar{S}_\Sigma(0) D_i, \\ &\quad 1 \leq i \leq N+M; \\ \bar{\Delta}_{ij} &= -D_i^T \bar{S}_\Sigma(0) D_j, \\ &\quad 1 \leq i, j \leq N+M \quad (i \neq j), \end{aligned}$$

$$\begin{aligned} \Pi_{0i} &= PH_i + A^T Q_i(\eta) + C_j^T \bar{S}_\Sigma(0) M_i \\ &\quad + C^T \bar{R}_{i\Sigma}^T(\eta, 0) + \bar{Q}_\Sigma(0) M_i, \quad 1 \leq i \leq N, \\ \Pi_{ji}(\eta) &= B_j Q_i(\eta) + D_j^T \bar{R}_{i\Sigma}^T(\eta, 0) \\ &\quad + D_j^T \bar{S}_\Sigma(0) M_i - \mathcal{X}(j) R_{ji}^T(\eta, -r_j), \\ &\quad 1 \leq j \leq N+M, \quad 1 \leq i \leq N; \end{aligned}$$

$$\begin{aligned} \Lambda_{ij}(\xi, \eta) &= H_i^T Q_j(\eta) + Q_i^T(\xi) H_j + M_i^T \bar{S}_\Sigma(0) M_j \\ &\quad + M_i^T \bar{R}_{j\Sigma}^T(\eta, 0) + \bar{R}_{i\Sigma}(\xi, 0) M_j \\ &\quad - \frac{\partial R_{ij}(\xi, \eta)}{\partial \xi} - \frac{\partial R_{ij}(\xi, \eta)}{\partial \eta}, \quad 1 \leq i, j \leq N. \end{aligned}$$

with

$$\begin{aligned} \bar{Q}_\Sigma(0) &= \sum_{k=1}^N Q_k(0), \quad \bar{R}_{i\Sigma}(\eta, 0) = \sum_{k=1}^N R_{ik}(\eta, 0), \\ \bar{S}_\Sigma(0) &= \sum_{k=1}^N S_k(0) + \sum_{k=N+1}^{N+M} \bar{S}_k. \end{aligned}$$

Using Proposition 2, we can conclude that the system is asymptotically stable if (9) holds and the LKF and its derivative satisfy (4) and (5) for some  $\varepsilon > 0$ .

##### B. Discretization of LKF

Similar to [8], in the following we will restrict the functions  $Q_i$ ,  $R_{ij}$  and  $S_i$  to be piecewise linear. Specially, divide the interval  $[-r_i, 0]$ ,  $i = 1, 2, \dots, N$  into  $N_i$  smaller intervals  $\mathcal{I}_{ip} = [\theta_{ip}, \theta_{i,p-1}]$  of equal length  $h_i = r_i/N_i$ , where  $\theta_{ip} = -ph_i$ .

Let

$$Q_i^{(p)}(\alpha) \stackrel{def}{=} Q_i(\theta_{ip} + \alpha h_i) = (1-\alpha)Q_i^p + \alpha Q_i^{p-1}; \quad (11)$$

$$S_i^{(p)}(\alpha) \stackrel{def}{=} S_i(\theta_{ip} + \alpha h_i) = (1-\alpha)S_i^p + \alpha S_i^{p-1}; \quad (12)$$

$$\begin{aligned} R_{ij}^{(pq)}(\alpha, \beta) &\stackrel{def}{=} R_{ij}(\theta_{ip} + \alpha h_i, \theta_{jq} + \beta h_j) \\ &= \begin{cases} (1-\alpha)R_{ij}^{pq} + \beta R_{ij}^{p-1,q-1} + (\alpha-\beta)R_{i,j}^{p-1,q} & \text{for } \alpha \geq \beta; \\ (1-\beta)R_{ij}^{pq} + \alpha R_{ij}^{p-1,q-1} + (\beta-\alpha)R_{i,j}^{p,q-1} & \text{for } \alpha < \beta; \end{cases} \end{aligned} \quad (13)$$

for  $0 \leq \alpha, \beta \leq 1, p = 1, 2, \dots, N_i, q = 1, 2, \dots, N_j; i = 1, 2, \dots, N, j = 1, 2, \dots, N$ .

This choice allows us to write the stability conditions (4) and (5) in the form of LMIs. The idea is very similar to that of [10].

*Remark 6:* Note that the discretized approach above for the system with multiple delays is different to the one in [8], where the corresponding discretization was carried out in the delay-parameter space  $Or_{NrN}$ . In this paper, however, this discretization was derived in the delay-parameter space  $Or_{1r_2 \cdots r_N}$  which is first presented in [12].

**C. LKF Condition**

The following lemma specifies the conditions for (4).

*Lemma 7:* The LKF  $V$  defined in (3), with piecewise linear matrices  $Q_i, R_{ij}$ , and  $S_i$  as defined in (11) to (13), satisfies (4) if

$$\begin{aligned} S_i^p &> 0, \quad p = 1, 2, \dots, N_i, \quad i = 1, 2, \dots, N, \\ \tilde{S}_k &\geq 0, \quad k = N + 1, N + 2, \dots, M + N, \end{aligned}$$

and

$$\begin{pmatrix} P & Q \\ Q^T & \mathcal{R} + \mathcal{S} \end{pmatrix} > 0 \quad (14)$$

are satisfied, where

$$\begin{aligned} Q &= (\tilde{Q}_1 \quad \tilde{Q}_2 \quad \cdots \quad \tilde{Q}_N), \\ \tilde{Q}_i &= (Q_i^0 \quad Q_i^1 \quad \cdots \quad Q_i^{N_i}), \\ \mathcal{R} &= (\tilde{R}_{ij})_{i=1 \rightarrow N}^{j=1 \rightarrow N} \quad (\tilde{R}_{ij}^T = \tilde{R}_{ji}), \\ \tilde{R}_{ij} &= (R_{ij}^{pq})_{p=1 \rightarrow N_i}^{q=1 \rightarrow N_j}, \\ \mathcal{S} &= \text{diag}(\tilde{S}_1 \quad \tilde{S}_2 \quad \cdots \quad \tilde{S}_N), \\ \tilde{S}_i &= \text{diag}\left(\frac{1}{h_i} S_i^0 \quad \frac{1}{h_i} S_i^1 \quad \cdots \quad \frac{1}{h_i} S_i^{N_i}\right). \end{aligned}$$

*Proof:* Since  $V_2$  in (3) is positive semi-definite, we only need to show that  $V_1$  is positive definite. This is similar to the one discussed in [10]. ■

**D. Derivative Condition of LKF**

The next lemma established the conditions for (5). For simplicity of expression, define a matrix sequence

$$\mathcal{S}(i) = \begin{cases} S_i^{N_i}, & i = 1, 2, \dots, N; \\ \tilde{S}_i, & i = N + 1, N + 2, \dots, N + M. \end{cases}$$

Let

$$\begin{aligned} \bar{Q}_\Sigma^0 &= \sum_{k=1}^N Q_k^0, & \bar{R}_{i\Sigma}^{p0} &= \sum_{k=1}^N R_{ik}^{p0}, \\ \bar{S}_\Sigma^0 &= \sum_{k=1}^N S_k^0 + \sum_{k=N+1}^{N+M} \tilde{S}_k. \end{aligned}$$

*Lemma 8:* The derivative  $\dot{V}$  of the LKF in (10), with  $Q_i, S_i$  and  $R_{ij}$  ( $i, j = 1, 2, \dots, N$ ) piecewise linear as expressed in (11) to (13), satisfies (5) if there exists a real matrix  $W = W^T$  such that

$$\begin{pmatrix} \bar{\Delta} & -Y^s & -Y^a \\ -Y^{sT} & \Gamma - M^T S_\Sigma^0 M & -F^a \\ -Y^{aT} & -F^a & 3(S_d - W) \end{pmatrix} > 0 \quad (15)$$

$$\begin{pmatrix} W & R_{da} \\ R_{da}^T & W \end{pmatrix} > 0, \quad (16)$$

where

$$\Gamma = R_d + S_d - W - (F^s + F^{sT}), \quad (17)$$

$$\bar{\Delta} = (\bar{\Delta}_{ij})_{i=0 \rightarrow N+M}^{j=0 \rightarrow N+M} \quad (\bar{\Delta}_{ij}^T = \bar{\Delta}_{ji}),$$

$$\bar{\Delta}_{00} = -A^T P - PA - C^T S_\Sigma^0 C - \bar{Q}_\Sigma^0 C - C^T \bar{Q}_\Sigma^{0T},$$

$$\bar{\Delta}_{0i} = -PB_i - C^T S_\Sigma^0 D_i - \bar{Q}_\Sigma^0 D_i + \mathcal{X}(i) Q_i^{N_i},$$

$$1 \leq i \leq N + M;$$

$$\bar{\Delta}_{ij} = -D_i^T S_\Sigma^0 D_j \quad (i \neq j), \quad 1 \leq i, j \leq M + N,$$

$$\bar{\Delta}_{ii} = \mathcal{S}(i) - D_i^T S_\Sigma^0 D_i, \quad 1 \leq i \leq N + M;$$

$$S_d = \text{diag}(S_{d1} \quad S_{d2} \quad \cdots \quad S_{dN}),$$

$$S_{di} = \text{diag}(S_{di}^1 \quad S_{di}^2 \quad \cdots \quad S_{di}^{N_i}),$$

$$S_{di}^p = \frac{1}{h_i^2} (S_i^{p-1} - S_i^p);$$

$$R_{ds} = (R_{dsij})_{i=1 \rightarrow N}^{j=1 \rightarrow N}, \quad R_{dsij} = (R_{dsij}^{pq})_{p=1 \rightarrow N_i}^{q=1 \rightarrow N_j},$$

$$\begin{aligned} R_{dsij}^{pq} &= \frac{1}{2} \left( \frac{1}{h_i} + \frac{1}{h_j} \right) (R_{ij}^{p-1, q-1} - R_{ij}^{pq}) \\ &\quad + \frac{1}{2} \left( \frac{1}{h_j} - \frac{1}{h_i} \right) (R_{ij}^{p, q-1} - R_{ij}^{p-1, q}); \end{aligned}$$

$$R_{da} = (R_{daij})_{i=1 \rightarrow N}^{j=1 \rightarrow N}, \quad R_{daij} = (R_{daij}^{pq})_{p=1 \rightarrow N_i}^{q=1 \rightarrow N_j},$$

$$R_{daij}^{pq} = \frac{1}{2} \left( \frac{1}{h_j} - \frac{1}{h_i} \right) (R_{ij}^{p-1, q-1} - R_{ij}^{p-1, q} - R_{ij}^{p, q-1} + R_{ij}^{pq});$$

$$Y^s = (Y_{ij}^s)_{i=0 \rightarrow N+M}^{j=1 \rightarrow N},$$

$$Y_{ij}^s = \begin{pmatrix} Y_{ij}^{s1} & Y_{ij}^{s2} & \cdots & Y_{ij}^{sN_j} \end{pmatrix},$$

$$Y_{0i}^{sp} = PH_i + \bar{Q}_\Sigma^0 M_i + C^T S_\Sigma^0 M_i - \frac{1}{h_i} (Q_i^{p-1} - Q_i^p)$$

$$+ \frac{1}{2} A^T (Q_i^p + Q_i^{p-1}) + \frac{1}{2} C^T (\bar{R}_{i\Sigma}^{p, 0T} + \bar{R}_{i\Sigma}^{p-1, 0T}),$$

$$Y_{ji}^{sp} = \frac{1}{2} B_j^T (Q_i^p + Q_i^{p-1}) + \frac{1}{2} D_j^T (\bar{R}_{i\Sigma}^{p, 0T} + \bar{R}_{i\Sigma}^{p-1, 0T})$$

$$- \frac{1}{2} \mathcal{X}(j) (R_{ji}^{p, N_j T} + R_{ji}^{p-1, N_j T}) + D_j^T S_\Sigma^0 M_i,$$

$$1 \leq i \leq N, 1 \leq j \leq N + M;$$

$$Y^a = (Y_{ij}^a)_{i=0 \rightarrow N+M}^{j=1 \rightarrow N},$$

$$Y_{ij}^a = \begin{pmatrix} Y_{ij}^{a1} & Y_{ij}^{a2} & \cdots & Y_{ij}^{aN_j} \end{pmatrix},$$

$$Y_{0i}^{ap} = \frac{1}{2} A^T (Q_i^p - Q_i^{p-1}) + \frac{1}{2} C^T (\bar{R}_{i\Sigma}^{p, 0T} - \bar{R}_{i\Sigma}^{p-1, 0T}),$$

$$Y_{ji}^{ap} = \frac{1}{2} B_j^T (Q_i^p - Q_i^{p-1}) + \frac{1}{2} D_j^T (\bar{R}_{i\Sigma}^{p, 0T} - \bar{R}_{i\Sigma}^{p-1, 0T})$$

$$- \frac{1}{2} \mathcal{X}(j) (R_{ji}^{p, N_j T} - R_{ji}^{p-1, N_j T}),$$

$$1 \leq j \leq N + M, 1 \leq i \leq N;$$

$$F^s = (F_{ij}^s)_{i=1 \rightarrow N}^{j=1 \rightarrow N}, \quad F_{ij}^s = (F_{ij}^{spq})_{i=0 \rightarrow N_i}^{j=1 \rightarrow N_j},$$

$$F_{ij}^{spq} = \frac{1}{2}H_i^T(Q_j^q + Q_j^{q-1}) + \frac{1}{2}M_i^T \sum_{k=1}^N (R_{jk}^{q,0T} + R_{jk}^{q-1,0T}),$$

$$F^a = (F_{ij}^a)_{i=1 \rightarrow N}^{j=1 \rightarrow N}, \quad F_{ij}^a = (F_{ij}^{apq})_{i=0 \rightarrow N_i}^{j=1 \rightarrow N_j},$$

$$F_{ij}^{apq} = \frac{1}{2}H_i^T(Q_j^q - Q_j^{q-1}) + \frac{1}{2}M_i^T (\bar{R}_{j\Sigma}^{q,0T} - \bar{R}_{j\Sigma}^{q-1,0T}). \quad (18)$$

*Proof:* The proof is similar to that of Proposition 5 in [8]. ■

*Remark 9:* It is noticed that (15) in Lemma 8 is evidently different with (16) of Theorem 3 in [11] by the dimension of  $\bar{\Delta}, Y^s$  and  $Y^a$ , the term  $\mathcal{S}(i)$  in  $\bar{\Delta}_{ii}$ , the multiplier  $\mathcal{X}(i)$  in  $\bar{\Delta}_{0i}$ ,  $Y_{ji}^{sp}$  and  $Y_{ji}^{ap}$ .

V. STABILITY OF THE WHOLE SYSTEMS

Combining Lemma 4, 7 and 8 together and applying Schur’s compensation we obtain the stability condition of the systems (1)-(2) as follows.

*Theorem 10:* The system expressed by (1)-(2) is uniformly asymptotically stable if there exist  $n$  by  $n$  matrix  $P = P^T$ ,  $n$  by  $m$  matrices  $Q_j^p$ ,  $m$  by  $m$  matrices  $R_{ij}^{pq}, S_i^p, p = 1, 2, \dots, N_i, q = 1, 2, \dots, N_j, i = 1, 2, \dots, N, j = 1, 2, \dots, N$ ; and  $m$  by  $m$  matrix  $W = W^T, X_i, i = 1, 2, \dots, N + M, Y_j, j = 1, 2, \dots, N$ , such that (9) (14) (15) and

$$\begin{pmatrix} \Delta & -Y^s & -Y^a & Z^T S_\Sigma^0 \\ -Y^{sT} & \Gamma & -F^a & M^T S_\Sigma^0 \\ -Y^{aT} & -F^{aT} & 3(R_d - W) & 0 \\ S_\Sigma^0 Z^T & S_\Sigma^0 M & 0 & S_\Sigma^0 \end{pmatrix} > 0 \quad (19)$$

are satisfied, with the notation defined in (15) to (15), (18) to (18) and

$$\Delta = (\Delta_{ij})_{i=0 \rightarrow N+M}^{j=0 \rightarrow N+M}$$

$$\Delta_{00} = -A^T P - PA - \sum_{k=1}^N (Q_k^0 C + C^T Q_k^{0T}),$$

$$\Delta_{0i} = -PB_i - \sum_{k=1}^N Q_k^0 D_i + \mathcal{X}(i) Q_i^{N_i}, \quad 1 \leq i \leq N + M;$$

$$\Delta_{ij} = 0 \quad (i \neq j), \quad 1 \leq i, j \leq M + N,$$

$$\Delta_{ii} = \mathcal{S}(i), \quad 1 \leq i \leq M + N;$$

and

$$Z = ( C \quad D_1 \quad D_2 \quad \dots \quad D_{N+M} ).$$

VI. NUMERICAL EXAMPLES

*Example 1:* Consider a system with two delays

$$\begin{aligned} \dot{x}(t) &= Ax(t) + By_2(t - r_2) + H \int_{-r_1}^0 y_1(t + \theta) d\theta, \\ y_1(t) &= C_1 x(t) + K_1 \int_{-r_1}^0 y_1(t + \theta) d\theta, \\ y_2(t) &= C_2 x(t) + D_1 y_1(t - r_1) + D_2 y_2(t - r_2) \\ &\quad - K_2 \int_{-r_1}^0 y_1(t + \theta) d\theta, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 \\ -0.5 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -2 & 0 \\ -1 & -1.45 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0.25 \\ -0.5 \\ 2.5 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$C_1 = ( 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 ),$$

$$C_2 = \begin{pmatrix} 0.2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}, \quad D_2 = 0.5I_2, \quad K_1 = 0.5, \quad K_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

This system come from [10], where it is given that the system is exponentially stable if and only if  $r_1 \in [0, 2\pi)$  and  $r_2 \in [0, 4.7388)$ . Although the above system and the example in [10] have the same characteristic equations, their stability is not equivalent. In fact, the additional dynamics

$$\det(I + K_1 \frac{1 - e^{-r_1 s}}{s}) = 0$$

is introduced in the difference-integral equation that the stability is a necessary condition for the whole systems, so that the stable region of  $r_1$  is determined by both the above equation and the characteristic equations. Noting that the additional dynamics is the characteristic equation of the integral equation

$$y(t) + 0.5 \int_{-r_1}^0 y(t + \sigma) d\sigma = 0. \quad (20)$$

Using the condition of Example 1 in [13], we obtained that (20) is exponentially stable if and only if  $r_1 \in [0, 2)$ . Connecting the stable interval  $[0, 2\pi)$  of  $r_1$ , we obtained that the whole systems is exponentially stable if and only if  $r_1 \in [0, 2)$  and  $r_2 \in [0, 4.7388)$ .

Applied Theorem 10 of this paper, for four given ratios  $1/\sqrt{5}, 1/\sqrt{2}, \sqrt{2}$  and  $\sqrt{5}$  of  $r_1/r_2$ , the upper bound of the delay  $r_2$  are computed by using MATLAB program with a bisection process, and the results are listed in the following table.

$r_1/r_2$	$1/\sqrt{5}$	$1/\sqrt{2}$	$\sqrt{2}$	$\sqrt{5}$
$(N_1, N_2)$	(1,2)	(1,2)	(2,1)	(2,1)
$r_{2max}$	4.4721	2.8284	1.4142	0.8944

From above table we are not difficult to find, when the ratios of  $r_1/r_2$  as  $1/\sqrt{5}, 1/\sqrt{2}, \sqrt{2}$  and  $\sqrt{5}$ , the supremum of  $r_1$  is 2. In fact, if the ratios of  $r_1/r_2$  is more than  $2/4.7388$ , the maximum estimate of  $r_2$  is completely restricted by the supremum 2 of  $r_1$ . However, for some fixed  $r_1 \in [0, 2)$  or setting  $r_1/r_2 \in$

[0, 2/4.7388), we can calculate that the maximum estimate of  $r_2$  can approximate the analytical limit. In the following we gave the results for  $r_1 = 1.9999$  and  $r_1/r_2 = 0.4$ .

$r_1=1.9999$				
$(N_1, N_2)$	(1,2)	(2,3)	(2,4)	Analytical
$r_{2max}$	4.7354	4.7381	4.7385	4.7388
$r_1/r_2=0.4$				
$(N_1, N_2)$	(1,2)	(1,3)	(2,4)	Analytical
$r_{2max}$	4.7354	4.7381	4.7385	4.7388

Example 2: Consider a two-dimensional system

$$\dot{x}(t) - D_0\dot{x}(t-r) = A_0x(t) + B_0x(t-r) + C_0 \int_{-\tau}^0 x(t+s)ds \quad (21)$$

with

$$A_0 = \begin{pmatrix} -a_1 & 0 \\ 0 & -a_2 \end{pmatrix}, B_0 = \begin{pmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{pmatrix}, C_0 = \begin{pmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{pmatrix}.$$

[14] and [15] discussed the stability of system for all  $r$  and allowable  $\tau$ . For the different parameters, it is verified by three cases.

C1:  $D_0 = 0, a_1 = a_2 = 1.5, b_1 = b_2 = 1, c_1 = 1, c_2 = 0.5$ .

C2:  $D_0 = 0, a_1 = 2, a_2 = 15, b_1 = 1, b_2 = 3, c_1 = 1, c_2 = 0.5$ .

C3:  $D_0 = \begin{pmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix}, a_1 = 2, a_2 = 15, b_1 = 1; b_2 = 3, c_1 = 1, c_2 = 0.5$ .

The system may be transformed into a standard coupled equation form as (1)-(2)

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + B\tilde{y}(t-r) + M \int_{-\tau}^0 \tilde{y}(t+s)ds, \\ \tilde{y}(t) &= \tilde{x}(t) + D_0\tilde{y}(t-r) - k \int_{-\tau}^0 \tilde{y}(t+s)ds. \end{aligned}$$

where  $k$  is a adjusting parameter and  $A = A_0 + kI, B = B_0 + (A_0 + kI)D_0, M = C_0 - (A_0 + kI)k$ .

While  $k = 0$ , applying the theorem of this paper, the stability of system are test for all  $r$  and allowable  $\tau$ . For the above three cases of parameters, the results are listed in the following table with the results in [15] and [14] together.

$\tau_{max}$	[14]	[15]	Theorem 1 for $N = 2$
C1	0.03	0.07	0.1138
C2	Not	1.1	1.2897
C3	Not	1	1.2303

For some fixed  $k(\neq 0)$  satisfied the condition (9), and let  $N = 2$ , the stability of system are test for all  $r$  and allowable  $\tau$ . For the case 1 and case 2 the results is same to the one with  $k = 0$ . For the case 3 the results are listed in the following table.

$k$	-0.5	-0.3	-0.1	0.1	0.3
C3	1.2299	1.2308	1.2306	1.2301	1.2298

The above data show, for the selection of suitable  $k$ , stability of the systems can be improved at a certain level.

### VII. CONCLUSION

The stability of a class of coupled differential-functional equations with multiple discrete and distributed delays is studied via a LKF approach. The discretized LKF method is used to render the problem into an LMI form that the discretization only involved to divide the known delay interval. The stability condition presented is independent with unknown delays, is dependent with known delays.

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