

A finite difference method for heat equation in the unbounded domain

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Abstract. The numerical method of the one-dimensional non-homogeneous heat equation on an unbounded domain is considered. Two exact artificial boundary conditions are applied on two artificial boundaries to limit the original problem onto a bounded computational domain. Then the finite difference method is developed by using the method of the reduction of order for the control equation and artificial boundary conditions. It is proved that the finite difference scheme is stable and convergent with the order 2 in space and order 3/2 in time under an energy norm. A non-homogeneous numerical example demonstrates the unconditional stability and the accuracy of the algorithm.

Keywords: non-homogeneous heat equation; unbounded domain; artificial boundary condition; finite difference method; error estimate.

1 Introduction

Heat equation arises from many fields, for examples, the heat transfer, fluid dynamics, astrophysics, finance or other areas of applied mathematics. For the problem on unbounded domains, Han and Huang (2002) in [1] and [2] derived an exact artificial boundary condition, obtained the corresponding numerical solution for heat equations by FEM and FDM in one-dimensional and two-dimensional cases, showed the effectiveness with numerical examples, but had no error estimate. Wu and Sun (2004) in [3] applied the artificial boundary condition in [1] to investigate the numerical solution of heat equation on the semi-infinite interval, proving the unconditional stability and convergence with order 2 in space and the order 3/2 in time under an energy norm.

In this paper, we consider the following Cauchy problem of one-dimensional non-homogeneous heat equation:

$$u_t - \sigma u_{xx} = f(x, t), \quad -\infty < x < +\infty, \quad 0 < t \leq T, \quad (1)$$

$$u(x, 0) = \phi(x), \quad -\infty < x < +\infty, \quad (2)$$

$$u(x, t) \rightarrow 0, \quad \text{when } |x| \rightarrow +\infty, \quad 0 \leq t \leq T. \quad (3)$$

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where σ is diffusion coefficient determining heat conduction rate, density and heat capacity of the material. The initial term $\phi(x)$ and the source term $f(x, t)$ have compact supports in space.

Introducing artificial boundaries

$$\Gamma_l = \{(x, t) \mid x = x_l, 0 \leq t \leq T\} \text{ and } \Gamma_r = \{(x, t) \mid x = x_r, 0 \leq t \leq T\},$$

such that $\text{supp}\{f(x, t)\} \subset [x_l, x_r] \times [0, T]$ and $\text{supp}\{\phi(x)\} \subset [x_l, x_r]$. According to the work of Han and Huang in [1] and Wu and Sun in [3], we can reduce the initial problem to an initial-boundary problem on a bounded domain:

$$\frac{\partial u}{\partial t} - \sigma \frac{\partial v}{\partial x} = f(x, t), \quad x_l < x < x_r, \quad 0 < t \leq T, \quad (4)$$

$$v - \frac{\partial u}{\partial x} = 0, \quad x_l < x < x_r, \quad (5)$$

$$u(x, 0) = \phi(x), \quad x_l < x < x_r, \quad (6)$$

$$v(x_l, t) = \frac{1}{\sqrt{\pi\sigma}} \int_0^t \frac{\partial u(x_l, \lambda)}{\partial \lambda} \frac{1}{\sqrt{t-\lambda}} d\lambda, \quad 0 < t \leq T, \quad (7)$$

$$v(x_r, t) = -\frac{1}{\sqrt{\pi\sigma}} \int_0^t \frac{\partial u(x_r, \lambda)}{\partial \lambda} \frac{1}{\sqrt{t-\lambda}} d\lambda, \quad 0 < t \leq T. \quad (8)$$

2 The finite difference scheme and corresponding error estimation

We discretize the above problem as follows. Divide the domain $[x_l, x_r] \times [0, T]$ into an $M \times N$ mesh with the spatial size $h = (x_r - x_l) / M$ in x direction and the time step size $\tau = T / N$ in t direction respectively, where M and N are positive integers. Grid points (x_i, t_n) are defined by $x_i = x_l + ih$ for $0 \leq i \leq M$ and $t_n = n\tau$ for $0 \leq n \leq N$. Moreover, denote U_i^n be the numerical solution of $u(x, t)$ at (x_i, t_n) and introduce the notations:

$$U_{i-\frac{1}{2}}^n = \frac{1}{2}(U_i^n + U_{i-1}^n), \quad U_i^{n-\frac{1}{2}} = \frac{1}{2}(U_i^n + U_i^{n-1}), \quad \delta_x U_{i-\frac{1}{2}}^n = \frac{1}{h}(U_i^n - U_{i-1}^n),$$

$$\delta_x^2 U_i^n = \frac{1}{h^2}(U_{i+1}^n - 2U_i^n + U_{i-1}^n), \quad \delta_t U_i^{n-\frac{1}{2}} = \frac{1}{\tau}(U_i^n - U_i^{n-1}),$$

$$f_{i-\frac{1}{2}}^n = \frac{1}{2}(f_i^n + f_{i-1}^n), \quad f_i^{n-\frac{1}{2}} = \frac{1}{2}(f_i^n + f_i^{n-1}), \quad \|U^n\|_E = \sqrt{h \sum_{i=1}^M (U_{i-\frac{1}{2}}^n)^2}.$$

Define the grid function: $u_i^n = u(x_i, t_n)$, $0 \leq i \leq M$, $n \geq 0$. The notations of v are similar.

Applying finite difference method for (4)-(8), we have

$$\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \sigma \delta_x v_{i-\frac{1}{2}}^{n-\frac{1}{2}} - f_{i-\frac{1}{2}}^{n-\frac{1}{2}} = r_{1,i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \leq i \leq M, n \geq 1, \quad (9)$$

$$v_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} = r_{2,i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \leq i \leq M, n \geq 1, \quad (10)$$

$$\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \sigma \delta_x v_{i-\frac{1}{2}}^{n-\frac{1}{2}} - f_{i-\frac{1}{2}}^{n-\frac{1}{2}} = r_{1,i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \leq i \leq M, n \geq 1, \quad (11)$$

$$v_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} = r_{2,i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \leq i \leq M, n \geq 1, \quad (12)$$

$$u_i^0 = \phi(x_i), \quad (13)$$

where $|r_{1,i-\frac{1}{2}}^{n-\frac{1}{2}}| \leq c_1(\tau^2 + h^2)$, $|r_{2,i-\frac{1}{2}}^{n-\frac{1}{2}}| \leq c_1 h^2$, $|s_1^{n-\frac{1}{2}}| \leq c_2 \tau^2$, $|s_2^{n-\frac{1}{2}}| \leq c_2 \tau^2$, $1 \leq i \leq M$, $n \geq 1$, and c_1 and c_2 are constants independent of τ and h .

Omitting the small error terms in (8)-(13), we construct a finite difference scheme:

$$\delta_t U_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \sigma \delta_x V_{i-\frac{1}{2}}^{n-\frac{1}{2}} - f_{i-\frac{1}{2}}^{n-\frac{1}{2}} = 0, \quad 1 \leq i \leq M, n \geq 1 \quad (14)$$

$$V_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x U_{i-\frac{1}{2}}^{n-\frac{1}{2}} = 0, \quad 1 \leq i \leq M, n \geq 1, \quad (15)$$

$$V_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\sigma}} [a_0 U_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \mathcal{U}_0^{k-\frac{1}{2}}], \quad n \geq 1 \quad (16)$$

$$V_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\sigma}} [a_0 U_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \mathcal{U}_M^{k-\frac{1}{2}}], \quad n \geq 1, \quad (17)$$

$$U_i^0 = \phi(x_i), \quad 0 \leq i \leq M. \quad (18)$$

Theorem 1. The finite difference scheme (14)-(18) is equivalent to the following scheme (19)-(23) without intermediate variable:

$$\frac{1}{2}(\delta_t U_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \delta_t U_{i+\frac{1}{2}}^{n-\frac{1}{2}}) - \sigma \delta_x^2 U_i^{n-\frac{1}{2}} = \frac{1}{2} \left(f_{i-\frac{1}{2}}^{n-\frac{1}{2}} + f_{i+\frac{1}{2}}^{n-\frac{1}{2}} \right), \quad (19)$$

$$\delta_t U_{\frac{1}{2}}^{n-\frac{1}{2}} + \frac{2\sigma}{h} \left[\frac{2}{\sqrt{\pi\sigma}} (a_0 U_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \mathcal{U}_0^{k-\frac{1}{2}}) - \delta_x U_{\frac{1}{2}}^{n-\frac{1}{2}} \right] = f_{\frac{1}{2}}^{n-\frac{1}{2}}, \quad (20)$$

$$\delta_t U_{M-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{2\sigma}{h} \left[\frac{2}{\sqrt{\pi\sigma}} (a_0 U_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \mathcal{U}_M^{k-\frac{1}{2}}) + \delta_x U_{M-\frac{1}{2}}^{n-\frac{1}{2}} \right] = f_{M-\frac{1}{2}}^{n-\frac{1}{2}}, \quad (21)$$

$$U_i^0 = \phi(x_i), \quad (22)$$

$$\text{where } a_m = \frac{1}{\sqrt{\tau}}(\sqrt{m+1} - \sqrt{m}), m = 1, 2, 3, \dots \quad (23)$$

We can obtain the stability and the convergence of the finite difference scheme using Gronwall inequality as follows by using the methods in [4] and [5].

Theorem 2. Let $\max_{x_j \leq x \leq x_r, 0 \leq t \leq T} |f(x, t)| \leq c_3$, c_3 be a positive constant. $\{U_i^n\}$ be the Solution of the difference scheme (14)-(18), $u(x, t)$ be the solution of problem (4)-(8). Then

$$\|U^n\|_E^2 \leq \frac{1}{1-\tau} \exp\left(\frac{2\tau}{1-\tau}\right) (\|U^0\|_E^2 + c_3^2 TL), n = 1, 2, \dots, T/\tau,$$

$$\|u^n - U^n\|_E^2 \leq \exp\left(\frac{2T}{2-\tau}\right) \frac{2c^2 T}{2-\tau} \left[\sqrt{\frac{\sigma T}{\pi}} + (\sigma + 2)L \right] (\tau^{\frac{3}{2}} + h^2)^2, \quad n=1, 2, \dots, [T/\tau],$$

where $c = \max\{c_1, c_2, c_3\}$.

3 Numerical example

We introduce the following definitions:

$$L^\infty\text{-error: } E_{h,\tau}^\infty = \max_{1 \leq i \leq M} |U_i^n - u_i^n|_{h,\tau}, \quad L^\infty\text{-order: } \log_2(E_{h,\tau}^\infty / E_{h/2,\tau}^\infty),$$

$$L^2\text{-error: } E_{h,\tau}^2 = \sqrt{h \sum_{i=1}^M [U_i^n - u_i^n]^2}_{h,\tau}, \quad L^2\text{-order: } \log_2(E_{h,\tau}^2 / E_{h/2,\tau}^2),$$

Example 1. Consider the following nonhomogeneous heat equation:

$$u_t - \sigma u_{xx} = (2\sigma - 1 - 4x^2)e^{-x^2-t}, \quad -\infty < x < +\infty, 0 < t \leq T,$$

$$u(x, 0) = e^{-x^2}, \quad -\infty < x < +\infty$$

$$u(x, t) \rightarrow 0, \text{ when } |x| \rightarrow +\infty, 0 \leq t \leq T.$$

which has the exact solution $u(x, t) = e^{-x^2-t}$. The support of $\phi(x)$ and $f(x, t)$ is compact on the computational domain $[x_l, x_r] = [-5, 5]$.

Table 1. Convergence w.r.t. τ of Example 1 for $\sigma = 1$ and $h = \tau^{3/4}$.

T=1						T=2				
M	N	L^∞ -error	order	L^2 -error	order	N	L^∞ -error	order	L^2 -error	order
56	10	4.7676e-3	-----	4.6564e-3	-----	20	2.2550e-3	-----	2.1077e-3	-----
95	20	1.6199e-3	1.5574	1.6091e-3	1.5330	40	7.5236e-4	1.5836	7.1586e-4	1.5579
159	40	5.7385e-4	1.4972	5.7163e-4	1.4931	80	2.6194e-4	1.5221	2.5128e-4	1.5104
267	80	2.0198e-4	1.5065	2.0202e-4	1.5006	160	9.1078e-5	1.5241	8.8160e-5	1.5111

Table 2. Convergence w.r.t. h of Example 1 for $\sigma = 1$ and $\tau = h^{4/3}$.

T=1						T=2					
M	N	L^∞ -error	order	L^2 -error	order	N	L^∞ -error	order	L^2 -error	order	
25	7	1.9906e-2	-----	2.2418e-2	-----	14	9.0971e-3	-----	9.7143e-3	-----	
50	17	5.6867e-3	1.8075	5.7087e-3	1.9734	34	2.5422e-3	1.8393	2.4764e-3	1.8710	
100	42	1.4219e-3	1.9998	1.4316e-3	1.9955	84	6.3418e-4	2.0093	6.1983e-4	1.9719	
200	107	3.5441e-4	2.0043	3.5782e-4	2.0003	214	1.5651e-4	2.0125	1.5461e-4	2.0032	

We can see in Table 1 that the order of convergence agrees with the theoretical order $3/2$ in time during the different time periods, since Theorem 2 gives $O(\tau^{3/2} + h^2) = O(\tau^{3/2})$ for $h = \tau^{3/4}$. Table2 illustrates the theoretical order 2 of convergence since Theorem 2 gives $O(\tau^{3/2} + h^2) = O(h^2)$ for $\tau = h^{4/3}$. In addition, the results when T=1 and T=2 show the unconditional stability of the proposed algorithm.

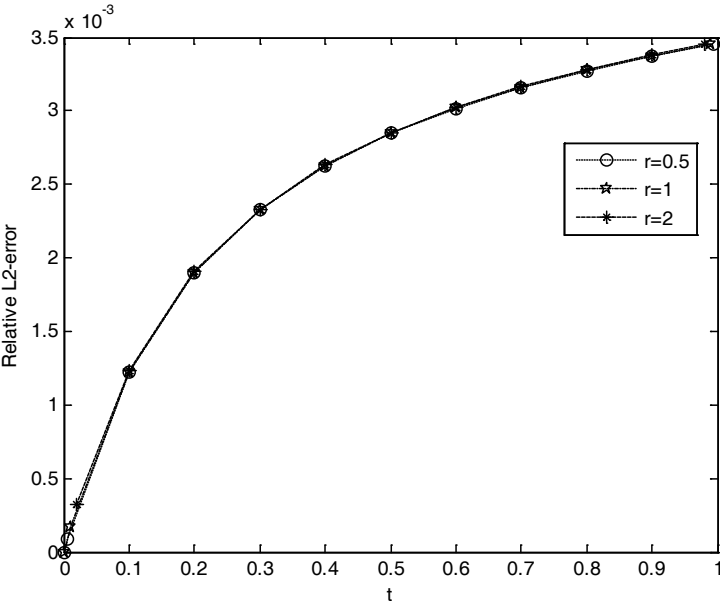


Figure 1. Relative L^2 -error of Example 1 during $0 \leq t \leq 1$ under different mesh ratios for $\sigma = 1$ and $M = 100$.

We can also see in Figure 1 that the numerical result is unconditionally stable during the computational time period, regardless of the mesh ratio $r = \frac{\sigma \tau}{h^2}$.

4 Conclusions

The numerical solution to the non-homogenous heat equation in the unbounded domain is considered. Similar to the numerical methods of non-homogenous heat equation in the half-unbounded domain in [1] and the involved homogenous heat equation in unbounded domain in [5], we convert the original problem to an initial-boundary value problem by using two exact artificial boundary conditions. Then the finite difference scheme is obtained by the method of reduction of order. The stability and the corresponding error estimate of (19)-(23) are given in Theorem 2. A non-

homogenous example demonstrates the unconditional stability and the accuracy of the proposed method.

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