The Sufficient Condition for the Original Function of the No Zero Entire Function is Infinite

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Key words: Entire function, Magnitude, Zero

Abstract: This paper proves that under certain conditions, so that there is an infinite number of zeros for $F(z) = \int_0^z \sum_{j=1}^m e^{\alpha_j(w)} dw$.

Introduction

The zero of the entire function is an important research topic in the theory of complex functions. We know that the entire function has infinitely many zeros and easy to get when the is very polynomial time. There must be an infinite number of zeros in the $F(z) = \int_0^z e^{\alpha(w)} dw$. And in general, for the special polynomial $\alpha(z)$, the zero point of is very much. For example, for

$Q(z) = \int_0^z e^{-w^n} dw \, (n \geq 2)$, there are (see literature). In the research of value distribution, the problem of the existence of the zero of the entire function, such as (which is as the entire function), is often confronted with the problem of the existence of the density of the zeros.

.Problem Presentation and main result

In 2004, C.C.Yang presented the following questions in the literature [4]:

If is a non-constant entire fuction, is a point in the area. So does have to have infinitely many zeros?

Background of the problem:

If is the entire function, then the entire function does not have zero and you can find the entire function $\alpha(z)$, so that does not have an zero function. So when the is a non-constant entire fuction, the number of zeros will be how to do?

We can easily get the above questions:

（1） $\alpha(z)$ is a non-constant polynomials, is a point in the area, then has infinitely many zeros;

（2） $\alpha(z)$ is a transcendental entire function, and $\rho(\alpha) \in \boxtimes /$, $F(z) = \int_{z_0}^z e^{\alpha(w)} dw$ also has infinitely many zeros.

We give a negative answer to the question above.

We have proved that under certain conditions $F(z)=\int_0^z e^{\alpha(w)}dw$, has a finite zero. So in what circumstances there are infinite number of zeros it? On this issue, we have the following results:

Lemmas: Set is not less than 2 of the positive integer, is a very integer function of m, ream

$$T^*(r)=\max_{1\le j\le m} T(r,e^{\alpha_j})(\forall r\in I) \quad, \quad \text{if} \quad \exists j,k\in\{1,2,\boxtimes,m\} \quad \text{and} \quad I\subset\boxtimes \quad, \quad \text{make} \quad mesI=+\infty \quad, \quad \text{and}$$

$T(r,e^{\alpha_j-\alpha_k})>\frac{1}{5}T^*(r)(\forall r\in I)$, then has infinitely many zeros.

Related lemmas

Lemmas 1[1] Set for the very number of the entire function, $f(z)=e^{\beta(z)}$, so

$$T(r,\beta)+T(r,\beta')=o(T(r,f))(r\to\infty).$$

Lemmas 2[3] set integer, is a linearly independent integral function of m, for $\forall z\in\boxtimes$, it has

$\max_{1\le j\le m}|g_j(z)|>0$, ream $T(r)=\frac{1}{2\pi}\int_0^{2\pi}u(re^{i\theta})-u(0)$, one of it $u(z)=\sup_{1\le j\le m}\log|g_j(z)|$, when

$g_{m+1}(z)=g_1(z)+g_2(z)+\boxtimes+g_m(z)$, it has $T(r)\le\sum_{j=1}^{m+1}N_{m-1}\left(r,\frac{1}{g_j}\right)+S(r)$, where meets enables to be

beyond the sub pure function, so $S(r)=o(T(r))(r\to+\infty;r\notin E,mesE<+\infty)$

Lemmas 3 [3] Under the condition of the lemma 2, for $\forall j,k\in\{1,2,\boxtimes,m\}$, it has

$$T\left(r,\frac{g_j}{g_k}\right)\le T(r)+O(1)(r\to\infty),$$

$$N\left(r,\frac{1}{g_j}\right)\le T(r)+O(1)(r\to\infty).$$

Lemmas 4 If there is a transcendental entire function $\partial(z)$, the polynomial $P(z)$ and the entire

function $\beta(z)$, fulfilled $F(z)=\int_0^z e^{\alpha(w)}dw=P(z)e^{\beta(z)}$, so $\rho(\alpha(z))=\rho(\beta(z))$.

Demondstrate of on both sides of the derivative.

Thus $T(r,e^{\alpha(z)})\le T(r,e^{\beta(z)})+o(T(r,e^{\beta(z)}))$

Therefore $T(r,e^{\alpha(z)})\le T(r,e^{\beta(z)})(1+o(1))$

From lemma 3 $\rho(\alpha)\le\rho(\beta)$

From formula $e^{\beta(z)}=\dfrac{e^{\alpha(z)}}{P'(z)+P(z)\beta'(z)}$

In the same way $T(r,e^{\beta(z)})\le T(r,e^{\alpha(z)})+o(T(r,e^{\beta(z)}))$

So $(1-o(1))T(r,e^{\beta(z)})\le T(r,e^{\alpha(z)})$

From lemmas 3 $\rho(\beta)\le\rho(\alpha)$

So $\rho(\alpha)=\rho(\beta)$

Proof of Theorem

Assuming that is at most only finite zeros, note that $F(0) = 0$, So there must be an entire function

and polynomial to make $\deg(P) \geq 1$, and $\int_0^z \sum_{j=1}^{m+1} e^{\alpha_j(w)} dw = P(z)e^{\beta(z)}$, (1)

On the sides of derivation $\sum_{j=1}^{m} e^{\alpha_j(z)} = \{P' + P\beta'\}e^{\beta(z)}$, (2)

Thus there $\sum_{j=1}^{m} e^{\alpha_j(z) - \beta(z)} - P'(z) = P(z)\beta'(z)$, (3)

Without loss of generality, can be set $\beta'(z) \neq 0$, is linearly independent, And used to replace.

There exist $r_0 > 0$, so when $|z| > r_0$, $|P'(z)| \geq 1$, make $T(z) = \frac{1}{2\pi}\int_0^z u(re^{i\theta})d\theta - u(0) (r \in I)$.

There is $u(z) = \log \max_{1 \leq j \leq m+1} |g_j(z)|$

Because of is a linear combination of admissible $m + 2$, Therefore, by the Cartan lemma

$$T(r) < N_{m]}\left(r, \frac{1}{P'}\right) + N_{m]}\left(r, \frac{1}{P\beta'}\right) + o\{T(r)\}(r \to \infty, r \notin E).$$

So is $(1 - o(1))T(r) \leq O(\log r) + o\left(T(r, e^{\beta})\right)$, (4)

From formula (2)

$$T(r, e^{\beta}) \leq T\left(r, \frac{1}{P' + P\beta'}\right) + T\left(r, \sum_{j=1}^{m+1} e^{\alpha_j}\right)$$

$$\leq T\left(r, \frac{1}{P' + P\beta'}\right) + \sum_{j=1}^{m+1} T\left(r, e^{\alpha_j}\right) + \log(m+1)$$

Thus the binding lemma

$$(1 - o(1))T\left(r, e^{\beta}\right) \leq \sum_{j=1}^{m+1} T\left(r, e^{\alpha_j}\right) \leq (m+1)T^*(r), \quad (5)$$

From and lemmas 3

$$\frac{1}{5}T^*(r) < T\left(r, e^{\alpha_j - \alpha_k}\right) \leq T(r) + O(1)(r \to \infty). \quad (6)$$

From formula $(4)(6)$

$$\frac{1 - o(1)}{5}T^*(r) \leq O(\log r) + o\left\{T\left(r, e^{\beta}\right)\right\},$$

So it has

$T^*(r) = O(\log r)$ contradiction,

So must have infinitely many zeros.

Conclusion

In the above theorem, we obtain a sufficient condition for the integral function of the form, such as $F(z) = \int_0^z \sum_{j=1}^{m} e^{\alpha_j(w)} dw$, to have infinitely many zeros. Under what conditions, we can obtain a neces-

sary and sufficient condition for the to have infinite zeros, and we will make further research on this problem.

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