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A Comparison of Distinct Consensus Measures for Group Decision Making with Intuitionistic Fuzzy Preference Relations

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Abstract

Intuitionistic fuzzy preference relation (IFPR), which express experts' preferences from the preferred, the non-preferred and the indeterminate aspects, has turned out to be an efficient tool in describing the rough and subjective opinions of experts. This paper focuses on the consensus measures for group decision making (GDM) in which all the experts use the IFPRs to express their preferences. Firstly, we give a brief analysis over the framework, the consistency checking process, and the selection process of intuitionistic fuzzy GDM. After that, two novel consensus measures, namely, the outranking flow based consensus measure and the ordinal consensus measure, are proposed to help an analyst to describe the degree of agreement among the experts in a group. In addition, an in-depth comparison is made from both theoretical and empirical points of view over our proposed consensus measures against the existing ones. Furthermore, a numerical example is given to show the difference among these distinct consensus measures. Finally, based on the ordinal consensus measure, a procedure is given to help the decision maker yield a final solution for GDM problems.

Keywords: Group decision making; intuitionistic fuzzy preference relation; consensus measure; intuitionistic fuzzy set; consensus reaching process.

1. Introduction

With the increasing global competition and uncertainty from exposure to a growing number of competitors, it is more and more essential for a firm or organization to make good decisions. To do so, many organizations prefer to form a team or group to take part in the key decision making process. Group decision making (GDM) has gained prominence in various fields, particularly in the important areas such as the financial, managerial, engineering and military fields. The benefits of GDM are quite numerous, such as better learning, accountability, fact screening, more knowledge, synergy, creativity, commitment and balanced risk propensity.[1] In this paper, we consider a GDM problem in which a set of experts $E=\{e_1,e_2,...,e_s\}$ are invited to make a choice from a number of alternatives $\Omega=\{\xi_1,\xi_2,\cdots,\xi_n\}$. When

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evaluating alternatives, in many cases, it may be difficult or impossible for experts to assess all the aspects of the candidate alternatives accurately. As a result, the experts often are only able to express their opinions roughly and subjectively through pairwise comparisons. Preference relation, which represents and stores the preference information of an expert over a set of alternatives in a matrix, has turned out to be a powerful tool in aiding decision making process.[2,3] There are many different types of preference relations, in which the most widely used ones are the multiplicative preference relation (MPR)[4], the fuzzy preference relation (FPR)[5], the interval-valued preference relation (IVPR)[6], the fuzzy linguistic preference relation (FLPR)[7], and the intuitionistic fuzzy preference relation (IFPR)[8]. The main difference between these preference relations appears on their distinct fundamental scales. The MPR uses a ratio scale, named 1/9-9 scale, to measure the intensities of pairwise comparisons over different alternatives, while the FPR employs a membership degree to express experts' preference information. The IVPR uses the subinterval of the unit interval [0, 1] to represent pairwise preferences. The FLPR expresses the intensities of pairwise comparisons in term of linguistic terms, such as "good", "a little good", and so on. The IFPR is quite different from all the above structures. The elements of the IFPR are represented by intuitionistic fuzzy numbers (IFNs)[9], each of which is characterized by a membership degree, a non-membership degree and a hesitancy degree. It is noted that all the elements in the MPR, the FPR, the IVPR and the FLPR are single values, which can only be used to describe the intensities of preferences but cannot depict the degrees of non-preferences. In many cases, however, it is very difficult for experts to determine accurate preference degrees, especially when some experts are not very familiar with a given GDM problem or there contains some incomplete information about the alternatives.[10] In such situations, the experts would prefer to express their opinions over the alternatives from three aspects, which are "preferred", "not preferred", and "indeterminate". The IFPR can be used to depict such preference information perfectly.

As IFPR can provide more flexible and comprehensive representation of experts' opinions than the other preference relation structures, it has gained many scholars' attentions. Szmit and Kacprzyk[11] first proposed the concept of IFPR. Later, Xu[8] gave a simplified notion for it and illustrated how to use it in decision making by a practical example concerning the assessment of a set of agroecological regions in Hubei Province, China. Xu and Liao[12] introduced the framework of the intuitionistic fuzzy AHP (IFAHP) method in which the experts' opinions are described in IFPRs, and applied the IFAHP method to global supplier management. Gong et al.[13] proposed some goal programming approaches to obtain the priority vector from an IFPR. Later, Wang[14] introduced a different linear goal programming model to obtain the normalized intuitionistic fuzzy priorities of an IFPR. After reviewing all the existing definitions of consistency for an IFPR, Liao and Xu[15] proposed a novel definition of multiplicative consistency to measure the consistency of an IFPR, based on which, a set of interesting fractional models were developed to derive the priorities of an IFPR. Considering that there are some IFPRs which are not consistent, Liao and Xu [16] proposed some automatic procedures to repair the inconsistent IFPRs. They[17] also investigated the multiplicative consistency of interval-valued IFPRs. All these achievements show that the IFPR is a hot and interesting research topic regarding to decision making.

For decision making with single IFPR, the fundamental issue is to yield priorities from an IFPR and then rank the alternatives according to the derived weights. However, for GDM with IFPRs, the situations are usually much harder owing to the complexity introduced by the conflicting opinions from experts. To better understand the GDM problem with IFPRs, Liao et al.[10] proposed a framework for GDM with IFPRs, and pointed out the difficulties which would take place in the process of GDM. Generally, a GDM problem with IFPRs consists of three sub-problems: the consistency checking process of each IFPR, the consensus checking process of the group, the selection process. Since GDM has many benefits and is much closer to practical decision making situations, many scholars have paid their attention to GDM with IFPRs and have achieved many fruitful results. After introducing the notion of IFPR, Xu[8] proposed a simple algorithm for GDM with IFPRs. Xu and Xia[18] developed some iterative procedures to improve the consistency of IFPRs in a group, which was based on the multiplicative consistency of an IFPR introduced in Ref. 19. Later, based on a more reasonable multiplicative consistency definition proposed in Ref. 15, Liao and Xu[20] developed two sorts of methods for GDM with IFPRs, i.e., aggregate individual priority vectors and aggregate individual IFPRs. All those works in Refs. 8, 18 and 20 were focused on the first sub-problem and the third sub-problem, but do not consider the second sub-problem, i.e., the consensus reaching process.

In fact, the consensus reaching process should be the most important process because only this process can guarantee that the final result be supported by all the group members despite their different opinions.[1] Kacprzyk and Fedrizzi[21] used a fuzzy-logic-based calculus of linguistically quantified propositions to measure the degree of consensus for a group in which the individuals' opinions are represented with FPRs. Palomares et al.[22] provided an overview and categorization of some existing consensus models for GDM in a fuzzy context and then presented a simulation-based analysis framework to study the performance of each consensus model. Herrera-Viedma et al.[23] made an overview of consensus models based on soft consensus measures for the MPRs, the FPRs and the FLPRs, but didn't mention the IFPRs. Based on the minimum cost and maximum return, Gong et al.[24] constructed some consensus models and gave their economic interpretation. In terms of the IFPRs, Szmidt and Kacprzyk[25] firstly paid attention to this issue and investigated the consensus of IFPRs by extending the idea of fuzzy consensus analysis based on α -cuts of the respective individual preference relations. After that, based on the similarity measure between IFPRs, Xu and Yager[26] developed some consensus analysis method for GDM with IFPRs. Recently, Liao et al.[10,27] introduced another consensus measure and developed an enhanced consensus reaching process. Since consensus measure is the way to represent the agreement degree among a group of experts, it is fundamentally important in finding the final solution for GDM with IFPRs. Although the consensus measures for GDM within the context of fuzzy set have been investigated by many authors, as reviewed above, the consensus checking process of GDM with IFPRs gained less importance in the literature. In addition, as far as we know, there is no paper doing a comparison analysis over different consensus measures for GDM with IFPRs. This paper is dedicated to discuss different kinds of consensus measures for GDM with IFPRs. Briefly speaking, the novelties of this paper can be summarized as follows:

- We propose two novel measures to describe the consensus of a group with IFPRs.
- We give an in-depth comparison between these different types of consensus measures from theoretical points of view.
- We test the applicability of these measures as well with a numerical example. The comparisons of these measures show that the proposed measures outperform than the exist measures.
- After that, an approach to intuitionistic fuzzy GDM with our proposed consensus measures is provided.

It should be noted that proposing two novel consensus measures is just one part of innovation of this paper. The most important significance of this paper is to make an overview and in-depth comparison of distinct consensus measures for GDM with IFPRs. To the best of our knowledge, this is the first paper to address this issue. It should also be noted that, although we focus on the consensus reaching process of intuitionistic fuzzy GDM in this paper, it does not mean that we ignore the significance of the other two sub-problems. To simplify the presentation, the consistency checking process and the selection process are assumed to be done with the method proposed in Ref. 10.

Based on this focus, the remainder of this paper is set out as follows: Section 2 describes the GDM problem with IFPRs and then makes some brief description on the consistency checking process and the selection process of intuitionistic fuzzy GDM. Two novel consensus measures are developed in Section 3. Section 4 firstly reviews the existing consensus measures for GDM with IFPRs and then makes some in-depth comparisons over all the six types of consensus measures of intuitionistic fuzzy GDM from both theoretical and empirical points of view. An ordinal consensus measure based intuitionistic fuzzy GDM procedure is given in Section 5. The paper ends in Section 6.

2. GDM with Intuitionistic Fuzzy Preference Information

2.1. Description of GDM with intuitionistic fuzzy preference relation

Intuitionistic fuzzy set (IFS)[28], which assigns to each element a membership degree, a non-membership degree and a hesitancy degree, has turned out to be a powerful tool to express vague and imprecise information. An IFS $\tilde{A}$ on X is a set of ordered triples, $\tilde{A}=\{(x,\mu_A(x),v_A(x))\,|\,x\in X\}$, where μ_A and v_A are the membership and non-membership functions mapping from X to [0,1] with the condition $0\le\mu_A+v_A\le 1$. For each $x\in X$, $\mu_A(x)$ represents its membership degree to $A\subseteq X$, and $v_A(x)$ gives the non-membership degree. The number $\pi_A(x)=1-\mu_A(x)-v_A(x)$ is called the hesitancy degree of x to $A\subseteq X$. The membership function of IFS is exactly the same as that in fuzzy set, while the non-membership function gives the opposite information from the negative point of view. The ordinary fuzzy set over X can be viewed as special IFSs with the non-membership function $v_A(x)=1-\mu_A(x)$. For convenience, Xu[9] called $\tilde{\alpha}_x=(\mu_x,v_x,\pi_x)$ an IFN. For example, let $\tilde{A}$ denote "the vote for resolution $x\in X$ ", then the IFN $\tilde{\alpha}_x=(0.5,0.3,0.2)$ can be interpreted as "the vote for resolution is 5 in favour, 3 against, and 2 abstentions". Szmidt and Kacprzyk[29] justified that π_x cannot be omitted when calculating the distance between two IFSs. For two IFNs $\tilde{\alpha}_x=(\mu_x,v_x,\pi_x)$ and $\tilde{\alpha}_y=(\mu_y,v_y,\pi_y)$, the normalized Hamming distance was defined as:[29]

$$d(\tilde{\alpha}_x,\tilde{\alpha}_y)=\frac{1}{2}(|\mu_x-\mu_y|+|v_x-v_y|+|\pi_x-\pi_y|) \quad (1)$$

and it satisfies $0\le d(\tilde{\alpha}_x,\tilde{\alpha}_y)\le 1$.

For a GDM problem, let $\Omega=\{\xi_1,\xi_2,...,\xi_n\}$ be the set of alternatives and $E=\{e_1,e_2,...,e_s\}$ be the set of experts who are associated with a weight vector $\lambda=(\lambda_1,\lambda_2,...,\lambda_s)^T$, where $\lambda_l>0, l=1,2,...,s$, and $\sum\lambda_l=1$. In practical GDM process, the weights of experts are determined subjectively or objectively in advance according to their expertise and experience. If there is no evidence to show significant differences among the experts or specific preference on some experts, the equal weights should be given. In the process of GDM, if the experts use IFNs to express their fuzzy and imprecise pairwise preferences, then some IFPRs are constructed. Suppose that the expert e_l provides his/her preference for the alternative ξ_i against the alternative ξ_j as $\tilde{r}_{ij}^{(l)}=(\mu_{ij}^{(l)},v_{ij}^{(l)},\pi_{ij}^{(l)})$ in which $\mu_{ij}^{(l)}$ denotes the degree that the alternative ξ_i is preferred to the alternative ξ_j, $v_{ij}^{(l)}$ indicates the degree that the alternative ξ_i is not preferred to the alternative ξ_j, and $\pi_{ij}^{(l)}=1-\mu_{ij}^{(l)}-v_{ij}^{(l)}$ is interpreted as the hesitancy degree. The IFPR $\tilde{R}^{(l)}=(\tilde{r}_{ij}^{(l)})_{n\times n}$ given by the expert e_l can be written as:

$$\tilde{R}^{(l)}=\begin{pmatrix}\tilde{r}_{11}^{(l)} & \tilde{r}_{12}^{(l)} & \cdots & \tilde{r}_{1n}^{(l)}\\ \tilde{r}_{21}^{(l)} & \tilde{r}_{22}^{(l)} & \cdots & \tilde{r}_{2n}^{(l)}\\ \vdots & \vdots & \ddots & \vdots\\ \tilde{r}_{n1}^{(l)} & \tilde{r}_{n2}^{(l)} & \cdots & \tilde{r}_{nn}^{(l)}\end{pmatrix} \quad (2)$$

where $\mu_{ij}^{(l)},v_{ij}^{(l)}\in[0,1]$, $\mu_{ij}^{(l)}+v_{ij}^{(l)}\le 1$, $\mu_{ij}^{(l)}=v_{ji}^{(l)}$, $\mu_{ii}^{(l)}=v_{ii}^{(l)}=0.5$, for all $i,j=1,2,...,n$, $l=1,2,...,s$.

Such a GDM problem is complicated owing to the conflicting opinions from experts. To solve this problem, Liao *et al.*[10] introduced a framework for GDM with IFPRs, which divided this complicated GDM problem into three processes, i.e., the consistency checking process of each individual IFPR, the consensus checking process of the group and the selection process (For details, please refer to Ref. 10).

2.2. The consistency checking and inconsistency repairing process for each IFPR

Consistency checking process is proposed to handle the conflicting opinions determined by the individual experts themselves. It is highly important because the lack of consistency in preference relations may lead to unreasonable conclusions. In order to check the consistency of an IFPR, we should first establish the way to measure the consistency of an IFPR. There are mainly two kinds of consistency for IFPR, namely, the additive consistency[14] and the multiplicative consistency[12,13,15,19]. After in-depth review and analyses over these distinct consistency definitions, Liao and Xu[15] proposed a general definition of multiplicative consistency for IFPR, and showed that this kind of consistency can be used as a standard to measure the consistency for an IFPR.

Definition 1.[15] An IFPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij}=(\mu_{ij},v_{ij},\pi_{ij})$ is multiplicative consistent if the multiplicative transitivity $\mu_{ij}\cdot\mu_{jk}\cdot\mu_{ki}=v_{ij}\cdot v_{jk}\cdot v_{ki}$ are satisfied for all $i,j,k=1,2,\cdots,n$.

Let $\tilde{\omega}=(\tilde{\omega}_1,\tilde{\omega}_2,\cdots,\tilde{\omega}_n)^T$ be a underlying intuitionistic fuzzy priority vector of an IFPR $\tilde{R}$, where $\tilde{\omega}_i=(\omega_i^\mu,\omega_i^v)$ $(i=1,2,\cdots,n)$ is an IFN such that $\omega_i^\mu,\omega_i^v\in[0,1]$ and $\omega_i^\mu+\omega_i^v\le 1$. ω_i^μ and ω_i^v indicate the membership and non-membership degrees of the alternative ξ_i as per a fuzzy concept of "importance", respectively. $\tilde{\omega}$ is said to be normalized if

$\sum_{j=1,j\neq i}^{n} \omega_j^\mu \leq \omega_i^v$, $\omega_i^\mu + n - 2 \geq \sum_{j=1,j\neq i}^{n} \omega_j^v$, for all $i = 1,2,\cdots,n$.[15] With $\tilde{\omega}$, a perfect multiplicative consistent IFPR $\tilde{R}^* = (\tilde{r}_{ij}^*)_{n\times n}$ can be established as:[15]

$$\tilde{r}_{ij}^* = (\mu_{ij}^*, v_{ij}^*) = \begin{cases} (0.5, 0.5) & if\ i = j \\ \left(\dfrac{2\omega_i^\mu}{\omega_i^\mu - \omega_i^v + \omega_j^\mu - \omega_j^v + 2}, \dfrac{2\omega_j^\mu}{\omega_i^\mu - \omega_i^v + \omega_j^\mu - \omega_j^v + 2}\right) & if\ i \neq j \end{cases} \quad (3)$$

Thus, a fractional programming model can be constructed to derive the priority vector $\tilde{\omega}$ from any a IFPR $\tilde{R}$ by minimizing the deviation between $\tilde{R}$ and its corresponding perfect multiplicative consistent IFPR $\tilde{R}^*$:[15]

Model 1 Min $J = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\varepsilon_{ij}^+ + \varepsilon_{ij}^- + \xi_{ij}^+ + \xi_{ij}^-\right)$

$$s.t.\ \begin{cases} \dfrac{2\omega_i^\mu}{\omega_i^\mu - \omega_i^v + \omega_j^\mu - \omega_j^v + 2} - \mu_{ij} - \varepsilon_{ij}^+ + \varepsilon_{ij}^- = 0, & i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n \\ \dfrac{2\omega_j^\mu}{\omega_i^\mu - \omega_i^v + \omega_j^\mu - \omega_j^v + 2} - v_{ij} - \xi_{ij}^+ + \xi_{ij}^- = 0, & i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n \\ \omega_i^\mu, \omega_i^v \in [0,1],\ \omega_i^\mu + \omega_i^v \leq 1, & i = 1,2,\cdots,n-1 \\ \sum_{j=1,j\neq i}^{n} \omega_j^\mu \leq \omega_i^v,\ \omega_i^\mu + n - 2 \geq \sum_{j=1,j\neq i}^{n} \omega_j^v, & i = 1,2,\cdots,n-1 \\ \varepsilon_{ij}^+ \geq 0, \varepsilon_{ij}^- \geq 0, \xi_{ij}^+ \geq 0, \xi_{ij}^- \geq 0, \varepsilon_{ij}^+ \cdot \varepsilon_{ij}^- = 0, \xi_{ij}^+ \cdot \xi_{ij}^- = 0, & i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n \end{cases}$$

where $\varepsilon_{ij} = \dfrac{2\omega_i^\mu}{\omega_i^\mu - \omega_i^v + \omega_j^\mu - \omega_j^v + 2} - \mu_{ij}$, $\xi_{ij} = \dfrac{2\omega_j^\mu}{\omega_i^\mu - \omega_i^v + \omega_j^\mu - \omega_j^v + 2} - v_{ij}$, $\varepsilon_{ij}^+ = \dfrac{|\varepsilon_{ij}| + \varepsilon_{ij}}{2}$, $\varepsilon_{ij}^- = \dfrac{|\varepsilon_{ij}| - \varepsilon_{ij}}{2}$, $\xi_{ij}^+ = \dfrac{|\xi_{ij}| + \xi_{ij}}{2}$, $\xi_{ij}^- = \dfrac{|\xi_{ij}| - \xi_{ij}}{2}$, $i, j = 1,2,\cdots,n;\ i \neq j$.

Observe that furnishing a perfect multiplicative consistent IFPR is somehow too strict for the experts, especially when the number of objects is very large. Thus, Liao and Xu[34] defined the acceptable multiplicative consistent IFPR, which satisfies

$$d(\tilde{R}, \tilde{R}^*) \leq \zeta \quad (4)$$

where ζ is the consistency threshold and $d(\tilde{R}, \tilde{R}^*)$ is the distance measure between the given IFPR $\tilde{R}$ and its corresponding perfect multiplicative consistent IFPR $\tilde{R}^*$ and can be calculated by $d(\tilde{R}, \tilde{R}^*) = \dfrac{1}{(n-1)(n-2)}\sum_{1\leq i<j<n}^{n}\left(\left|\mu_{ij} - \mu_{ij}^*\right| + \left|v_{ij} - v_{ij}^*\right| + \left|\pi_{ij} - \pi_{ij}^*\right|\right)$.

If some IFPRs are not of acceptable consistency, the analyst who coordinates the group decision process would return the inconsistent IFPRs to the corresponding experts immediately and asks them to repair the inconsistent ones. If an expert refuses to modify his/her inconsistent IFPR, this expert should be removed from the expert group since his/her preference values are self-contradict. If an expert is willing to revise his/her inconsistent IFPRs $\tilde{R}^p = (\tilde{r}_{ij}^p)_{n\times n}$, then $\tilde{R}^p$ can be modified into $\tilde{R}^{p+1} = (\tilde{r}_{ij}^{p+1})_{n\times n}$ by the following iterative formulas:

$$\mu_{ij}^{p+1} = \left(\mu_{ij}^p\right)^{1-p\eta} \cdot \left(\mu_{ij}^{p*}\right)^{p\eta},\ v_{ij}^{p+1} = \left(v_{ij}^p\right)^{1-p\eta} \cdot \left(v_{ij}^{p*}\right)^{p\eta}$$
$$i, j = 1,2,\cdots,n \quad (5)$$

where $\tilde{R}^{p*} = (\tilde{r}_{ij}^{p*})_{n\times n}$ with $\tilde{r}_{ij}^{p*} = (\mu_{ij}^{p*}, v_{ij}^{p*}, \pi_{ij}^{p*})$ is the corresponding multiplicative consistent IFPR of $\tilde{R}^p$, p is the number of iteration and η is the step size satisfying $0 \leq p\eta \leq 1$. Liao *et al.*[10] showed that this iteration process is convergent.

2.3. The selection process

The selection process is operated when the group of experts reach acceptable consensus. This process is used to find the final result. One question aroused here is how to aggregate these individual priorities or individual judgments together. Liao and Xu[30] proved that only the simple intuitionistic fuzzy weighted geometric (SIFWG) operator is reasonable to synthesize the individual IFPRs because only this method can guarantee that the fused IFPR is still of acceptable consistency when all individual IFPRs are of acceptable consistency. Suppose that there are s^* experts E_l ($l = 1,2,\cdots,s^*$) who reach the final consensus after the consensus reaching process with the underlying priority vectors $\tilde{\omega}^{(l)} = (\tilde{\omega}_1^{(l)}, \tilde{\omega}_2^{(l)}, \cdots, \tilde{\omega}_n^{(l)})^T$, $l = 1,2,\cdots,s^*$. To select the best alternative(s), the priority vectors $\tilde{\omega}^{(l)}$ ($l = 1,2,\cdots,s^*$) are aggregated by the SIFWG operator, i.e.:

$$\tilde{\omega}_i = (\omega_i^\mu, \omega_i^v) = \left(\prod_{l=1}^{s^*}\left(\omega_i^{\mu(l)}\right)^{\lambda_j}, \prod_{l=1}^{s^*}\left(\omega_i^{v(l)}\right)^{\lambda_j}\right) \quad (6)$$

Comparing the overall intuitionistic fuzzy weights $\tilde{\omega}_i$ ($l = 1,2,\cdots,s^*$), we can select the best alternative A_{i^*}, where

$$i^* = \arg\max_{i=1,2,\cdots,n}\{\tilde{\omega}_i\} \quad (7)$$

3. Two Novel Consensus Measures for GDM with Intuitionistic Fuzzy Preference Relations

Consensus is the most important issue in GDM with IFPRs owing to the fact that it is a pathway to a true group decision being supported by all the experts despite their different opinions. Traditionally, consensus is defined as the full and unanimous agreement among all experts regarding to all possible alternatives. However, as to practical decision making process, such a strict consensus is hard to achieve and thus in most cases, consensus is said to be attained if the expert community has largely solved the problems of the domain. The consensus can be defined as a state of mutual agreement among members of a group where all opinions have been heard and addressed to the satisfaction of the group.[31] Such a consensus is not to be enforced through negotiations or bargaining process, but to emerge after exchanges of opinions among the members of a group.[1] The consensus reaching process is an iterative process where the experts accept to change their opinions following the advice given by the figure of a moderator. The moderator can be seen as an analyst who does not take part in the communication process but knows the degree of consensus in each round of iteration. As a moderator, the most essential task he/she should do is to measure the consensus. Up to now, there are several different consensus measures for intuitionistic fuzzy GDM proposed by different scholars. In the following, we first propose two new consensus measures to help the moderator or analyst to assess the degree of agreement among the experts in a group, and then compare them with the existing consensus measures for GDM with IFPRs.

3.1. The outranking flow based consensus measure

The outranking flow of a preference relation was first proposed in the PROMETHEE method[32]. Liao and Xu[33] investigated the PROMETHEE method within the context of intuitionistic fuzzy circumstances. The intuitionistic fuzzy PROMETHEE method is proposed to handle the situation in which the incomparability takes places in most pairwise comparisons. The critical concepts of this method are the intuitionistic fuzzy positive outranking flow, the intuitionistic fuzzy negative outranking flow and the intuitionistic fuzzy net outranking flow. For an IFPR $\tilde{R}^{(l)}$ furnished by expert e_l, the intuitionistic fuzzy positive outranking flow for the alternative ξ_i is in the mathematical term of

$$\varphi_i^{(l)+} = \frac{1}{n-1} \mathop{\oplus}_{j=1, j\neq i}^{n} \tilde{r}_{ij}^{(l)} \tag{8}$$

and the intuitionistic fuzzy negative outranking flow for the alternative ξ_i is

$$\varphi_i^{(l)-} = \frac{1}{n-1} \mathop{\oplus}_{j=1, j\neq i}^{n} \tilde{r}_{ji}^{(l)} \tag{9}$$

The intuitionistic fuzzy positive outranking flow describes how the alternative ξ_i is outranking all the others. It is its power character. The higher $\varphi_i^{(l)+}$, the better the alternative ξ_i. The intuitionistic fuzzy negative outranking flow shows how the alternative ξ_i is outranked by all the others. It is its weakness character. The lower $\varphi_i^{(l)-}$, the better the alternative ξ_i. From this point of view, we can use the outranking flows to represent the overall value of each alternative. Therefore, the distance between any two experts e_l and e_m can be defined as

$$d(e_l, e_m) = \frac{1}{2n} \sum_{i=1}^{n} \left(d\left(\varphi_i^{(l)+}, \varphi_i^{(m)+}\right) + d\left(\varphi_i^{(l)-}, \varphi_i^{(m)-}\right) \right) \tag{10}$$

With the normalized Hamming distance shown as Eq. (1), Eq. (10) can be further expressed as

$$\begin{aligned} d(e_l, e_m) = \frac{1}{4n} \sum_{i=1}^{n} (& |\varphi_i^{\mu(l)+} - \varphi_i^{\mu(m)+}| + |\varphi_i^{v(l)+} - \varphi_i^{v(m)+}| \\ & + |\varphi_i^{\pi(l)+} - \varphi_i^{\pi(m)+}| + |\varphi_i^{\mu(l)-} - \varphi_i^{\mu(m)-}| \\ & + |\varphi_i^{v(l)-} - \varphi_i^{v(m)-}| + |\varphi_i^{\pi(l)-} - \varphi_i^{\pi(m)-}|) \end{aligned} \tag{11}$$

Since $0 \le d\left(\varphi_i^{(l)+}, \varphi_i^{(m)+}\right) \le 1$, $0 \le d\left(\varphi_i^{(l)-}, \varphi_i^{(m)-}\right) \le 1$, it is obvious that $0 \le d(e_l, e_m) \le 1$.

Considering the consensus is a state of mutual agreement among the members in a group, the outranking flow based consensus measure for a group of experts with IFPRs can be proposed.

Definition 2. For a GDM problem, suppose that the expert e_l provides an IFPR $\tilde{R}^{(l)} = \left(\tilde{r}_{ij}^{(l)}\right)_{n\times n}$. Then the consensus degree of such a group can be defined as:

$$CM_1 = 1 - \max_{l,m=1,2,\cdots,n} \{d(e_l, e_m)\} \tag{12}$$

where $d(e_l, e_m)$ is the distance between the experts e_l and e_m defined as Eq. (11).

Since $0 \le d(e_l, e_m) \le 1$, it is clear $0 \le CM_1 \le 1$. This outranking flow based consensus measure is easy to obtain. After given the consensus threshold, the analyst can check whether the group reaches the consensus or not. If not, then some consensus reaching process should be employed.

3.2. The ordinal consensus measure

Up to now, all the consensus measures for intuitionistic fuzzy GDM are based on the preferences given by experts. However, it is also reasonable to define the consensus based on the ranking of alternatives. As there are many methods to derive the priorities from each IFPR given by each individual expert, it is easy to obtain the ranks from each expert. Meanwhile, we can also derive the overall IFPR by the aggregation operators and then yield the group ranks for alternatives. In such a situation, the consensus can be defined as the differences between the orders of alternatives derived from group IFPR and those from individual IFPRs. Mathematically, suppose $o_i^{(l)}$ be the rank of the i th alternative from the l th expert, o_i^G be the rank of the i th alternative derived by the group. Then, the ordinal consensus measure for each expert can be defined as

$$CM_2^{(l)} = \sum_{i=1}^{n} \left(1 - \frac{|o_i^G - o_i^{(l)}|}{n-1} \right) \tag{13}$$

Therefore, the consensus measure of the group can be defined.

Definition 3. For a GDM problem, suppose that the expert e_l provides an IFPR $\tilde{R}^{(l)} = \left(\tilde{r}_{ij}^{(l)}\right)_{n \times n}$. Then the ordinal consensus degree of such a group can be defined as:

$$CM_2 = \frac{1}{s}\sum_{l=1}^{s} CM_2^{(l)} = \frac{1}{s}\sum_{l=1}^{s}\sum_{i=1}^{n} \left(1 - \frac{|o_i^G - o_i^{(l)}|}{n-1} \right) \tag{14}$$

This kind of consensus measure considers the relative positions of alternatives derived from individual experts and the group. There is no need to calculate the distance or similarity degrees between any pair of alternatives. It is easy to be understood by experts or other relevant persons. We can use Model 1 to derive the ranks of alternatives from individual IFPRs. Meanwhile, we can use the SIFWG operator to aggregate all individual IFPRs into an overall IFPR and then use Model 1 to derive the ranks of alternatives for the group.

4. Comparison with Distinct Consensus Measures for Intuitionistic Fuzzy GDM

In the above section, we introduce two novel consensus measures for intuitionistic fuzzy GDM. Since consensus measure plays a critical role in finding the agreeable solution for a GDM problem, in this section, we first review the existing consensus measures for GDM with IFPRs. After that, we make some comparison analyses over these distinct consensus measures.

4.1. Review on distinct consensus measures for GDM with IFPRs

4.1.1. The α -cut based consensus measure

The first paper regarding to consensus measure of intuitionistic fuzzy GDM was published by Szmit and Kacprzyk[25]. This type of consensus measure for IFPRs was based on the α -cut and motivated by the idea of Spillman *et al.*[34] on consensus of fuzzy relations. In this approach, they first divided each IFPR into two separate FPRs and then implemented Spillman *et al.*'s method over these two kinds of FPRs to derive two consensus degrees which are taken as the upper and lower bounds of the final interval-valued consensus degree of the IFPRs. Hence, with α -cut based consensus measure, the final consensus degree is represented by an interval. The algorithm for calculating the α -cut based consensus measure can be summarized as follows (For more details, please refer to Ref. 25).

Algorithm 1

Step 1: Divided each IFPR $\tilde{R}^{(l)}$ with $\tilde{r}_{ij}^{(l)} = (\mu_{ij}^{(l)}, v_{ij}^{(l)}, \pi_{ij}^{(l)})$ into two separate preference relations $U^{(l)} = (\mu_{ij}^{(l)})_{n \times n}$ and $\Delta^{(l)} = (\delta_{ij}^{(l)})_{n \times n}$ with $\delta_{ij}^{(l)} = \mu_{ij}^{(l)} + \pi_{ij}^{(l)}$, then go to the next step.

Step 2: Reduce each matrix $U^{(l)}$ and $\Delta^{(l)}$ into the α -cut preference relation $U^{\alpha(l)}$ and $\Delta^{\alpha(l)}$, where

$$\mu_{ij}^{\alpha(l)} = \begin{cases} 1 & if \quad \mu_{ij}^{\alpha(l)} \ge \alpha \\ 0 & if \quad \mu_{ij}^{\alpha(l)} < \alpha \end{cases}, \quad \delta_{ij}^{\alpha(l)} = \begin{cases} 1 & if \quad \delta_{ij}^{\alpha(l)} \ge \alpha \\ 0 & if \quad \delta_{ij}^{\alpha(l)} < \alpha \end{cases} \tag{15}$$

then go to the next step.

Step 3: For each $U^{(l)}$ and α, calculate the α-consensus matrix $C_{\min,\alpha}$ by

$$C_{\min,\alpha} = \begin{cases} A(U^{\alpha(l)}, U^{\alpha(k)}) & for\ l \neq k \\ 0 & otherwise \end{cases} \tag{16}$$

where

$$A(U^{\alpha(l)}, U^{\alpha(k)}) = \frac{tr\left(U^{\alpha(l)}, \left(U^{\alpha(k)}\right)^T\right)}{tr\left(U^{\alpha(l)}\left(U^{\alpha(l)}\right)^T\right) + tr\left(U^{\alpha(k)}\left(U^{\alpha(k)}\right)^T\right) - tr\left(U^{\alpha(l)}\left(U^{\alpha(k)}\right)^T\right)} \tag{17}$$

with $tr(\cdot)$ and $(\cdot)^T$ denote the usual trace and transpose operators, respectively. For each $\Delta^{(l)}$ and α, calculate α-consensus matrix $C_{\max,\alpha}$ with Eq. (16) and Eq. (17). Then go to the next step.

Step 4: For each α, compute $K_{\min,\alpha}$ and $K_{\max,\alpha}$ by

$$K_{\min,\alpha} = \frac{tr\left(C_{\min,\alpha}\left(C_{\min,\alpha}\right)^T\right)}{n(n-1)}$$
$$K_{\max,\alpha} = \frac{tr\left(C_{\max,\alpha}\left(C_{\max,\alpha}\right)^T\right)}{n(n-1)} \tag{18}$$

Go to the next step.

Step 5: Use the trapezoidal rules shown as Eq. (19) for numerical integration to find the total consensus $K_{\min}$ and $K_{\min}$. The final consensus of the group is $CM_3 = [K_{\min}, K_{\max}]$, where

$$K_{\min} = 0.1(0.1 + 2K_{\min,0.2} + 2K_{\min,0.4} + 2K_{\min,0.6} + 2K_{\min,0.8} + K_1)$$
$$K_{\max} = 0.1(0.1 + 2K_{\max,0.2} + 2K_{\max,0.4} + 2K_{\max,0.6} + 2K_{\max,0.8} + K_1) \tag{19}$$

4.1.2. The similarity based consensus measure

The similarity based consensus measure was also proposed by Szmit and Kacprzyk[35]. It was based on a new definition of similarity between two intuitionistic fuzzy sets. For any a IFN $\tilde{\alpha}_x = (\mu_x, v_x, \pi_x)$, its complement set $\tilde{\alpha}_x^c$ can be defined as $\tilde{\alpha}_x^c = (v_x, \mu_x, \pi_x)$. Then, the similarity measure for two IFNs $\tilde{\alpha}_x = (\mu_x, v_x, \pi_x)$ and $\tilde{\alpha}_y = (\mu_y, v_y, \pi_y)$ was introduced as

$$Sim(\tilde{\alpha}_x, \tilde{\alpha}_y) = d(\tilde{\alpha}_x, \tilde{\alpha}_y^c) - d(\tilde{\alpha}_x, \tilde{\alpha}_y) \tag{20}$$

where $d(\tilde{\alpha}_x, \tilde{\alpha}_y)$ is the hamming distance given as Eq. (1).

With Eq. (20), the similarity between any two experts e_l and e_m is given as

$$Sim(e_l, e_m) = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Sim(\tilde{r}_{ij}^{(l)}, \tilde{r}_{ij}^{(m)}) \tag{21}$$

Thus, the consensus degree of a group of experts whose preferences are given in IFPRs can be defined as

$$CM_4 = \frac{1}{s(s-1)} \sum_{l=1}^{s-1} \sum_{m=l+1}^{s} Sim(e_l, e_m) \tag{22}$$

4.1.3. The aggregation based consensus measure

Besides the above mentioned similarity based consensus measures, Xu and Yager[26] also proposed a consensus measure which was based on the aggregated IFPR and another different kind of similarity measure. We name this type of consensus measure as the aggregation based consensus measure. The main idea of this type of consensus measure is firstly deriving the overall IFPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij} = (\mu_{ij}, v_{ij}, \pi_{ij})$, where

$$\mu_{ij} = \sum_{l=1}^{s} \lambda_l \mu_{ij}^{(l)},\ v_{ij} = \sum_{l=1}^{s} \lambda_l v_{ij}^{(l)},\ \pi_{ij} = \sum_{l=1}^{s} \lambda_l \pi_{ij}^{(l)}$$
$$\mu_{ii} = v_{ii} = 0.5,\ \pi_{ii} = 0.5,\ i, j = 1, 2, \cdots, n \tag{23}$$

Then, the similarity degree of each individual IFPR $\tilde{R}^{(l)}$ to the overall IFPR $\tilde{R}$ is calculated as

$$Sim(\tilde{R}^{(l)}, \tilde{R}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Sim(\tilde{r}_{ij}^{(l)}, \tilde{r}_{ij}) \tag{24}$$

where

$$Sim(\tilde{r}_{ij}^{(l)}, \tilde{r}_{ij}) = \begin{cases} 0.5 & if\ \tilde{r}_{ij}^{(l)} = \tilde{r}_{ij} = \tilde{r}_{ij}^{c} \\ \dfrac{d(\tilde{r}_{ij}^{(l)}, \tilde{r}_{ij}^{c})}{d(\tilde{r}_{ij}^{(l)}, \tilde{r}_{ij}) + d(\tilde{r}_{ij}^{(l)}, \tilde{r}_{ij}^{c})} & otherwise \end{cases} \tag{25}$$

Thus, the consensus of the group is

$$CM_5 = \frac{1}{s} \sum_{l=1}^{s} Sim(\tilde{R}^{(l)}, \tilde{R}) \tag{26}$$

4.1.4. The distance based consensus measure

Recently, Liao *et al.*[10] proposed a quite interesting consensus measure for the GDM with IFPRs. This consensus measure was based on the underlying normalized intuitionistic fuzzy priority vector $\tilde{\omega}^{(l)} = (\tilde{\omega}_1^{(l)}, \tilde{\omega}_2^{(l)}, \cdots, \tilde{\omega}_n^{(l)})^T$ of each expert e_l

($l = 1, 2, \cdots, s$) derived by Model 1. For experts e_l and e_m, the distance measure between these two experts is

$$d(e_l, e_m) = \frac{1}{n}\sum_{i=1}^{n} d\left(\tilde{\omega}_i^{(l)}, \tilde{\omega}_i^{(m)}\right)$$
$$= \frac{1}{2n}\sum_{i}^{n}\left(\left(\left|\omega_i^{\mu(l)} - \omega_i^{\mu(m)}\right| + \left|\omega_i^{v(l)} - \omega_i^{v(m)}\right| + \left|\omega_i^{\pi(l)} - \omega_i^{\pi(m)}\right|\right)\right) \tag{27}$$

where $d\left(\tilde{\omega}_i^{(l)}, \tilde{\omega}_i^{(m)}\right)$ is the normalized Hamming distance between the IFNs $\tilde{\omega}_i^{(l)}$ and $\tilde{\omega}_i^{(m)}$. Based on this distance measure, the consensus degree among a group can be defined as:

$$CM_6 = 1 - \max_{l,m=1,2,\cdots,n}\{d(e_l, e_m)\} \tag{28}$$

4.2. Comparison of distinct consensus measures for GDM with IFPRs

As presented above, there are six different types of consensus measures for intuitionistic fuzzy GDM. All these consensus measures are represented by a number in the unit interval [0,1] except the α -cut based consensus measure CM_3, which uses an interval to describe the consensus degree of a group. Among these different consensus measures, only the second measure CM_2 proposed in this paper is based on the positions of alternatives, while all the other consensus measures are based on the preferences or the priorities of alternatives.

The outranking flow based consensus measure CM_1 and the similarity based consensus measure CM_4 are much simpler than the other four consensus measures because there is no need to calculate the underlying priorities from each individual IFPR and the overall IFPR and to do the aggregation calculation for these two consensus measures. On the contrary, for the ordinal consensus measure CM_2 and the distance based consensus measure CM_6, we have to calculate the underlying priorities from each individual IFPR and the overall IFPR; for the α -cut based consensus measure CM_3, we have to compute the trace of the preference relations; for the aggregation based consensus measure CM_5, we have to conduct aggregation process.

Comparing the similarity based consensus measure CM_4 with the aggregation based consensus measure CM_5, both of these two measures use the idea of similarity between two IFPRs; however, the similarity measures they used, shown as Eq. (20) and Eq. (25) respectively, are quite different. In addition, the former does not need to find the aggregated IFPR, but the latter method cannot work without such a collective IFPR. That is to say, if we want to calculate the consensus degree using CM_5, we should conduct the aggregation process first, which would definitely add the calculation complexity. Meanwhile, introducing such a collective IFPR would certainly add some residual errors, which may make the result be inefficient as well. Furthermore, it is observed that Eq. (25) is a fractional formula, which may also increase the invalidation of the final result. Thus, from this point of view, the similarity based consensus measure CM_4 should be more efficient than the aggregation based consensus measure CM_5.

Let us further compare the outranking flow based consensus measure CM_1 with the distance based consensus measure CM_6. Both of these two measures are based on the Hamming distance between the IFNs shown as Eq. (1). The difference between these two methodologies is: the first method calculates the distance between the outranking flows over the alternatives, while the second one compute the distance between the underlying priorities over the alternatives. Consequently, if we want to obtain the consensus degree through the consensus measure CM_6, we have to derive the underlying priorities for each alternative with respect to different experts. That is to say, the consensus measure CM_6 is much complicated than the outranking flow based consensus measure CM_1.

Among all these six consensus measures, the first one CM_1 and the sixth one CM_6 need to compute the distance between any two IFNs; the second one CM_2 and the sixth one CM_6 need to derive the underlying priorities; the third one CM_3 is the most complicated one owing to the fact that it needs to do much more calculation; the fourth one CM_4 and the fifth one CM_5 are both on the basis of similarities between two IFNs.

The following example adopted from Szmidt and Kacprzyk's paper[25] is to illustrate the differences between these distinct consensus measures for GDM with IFPRs.

Example 1. There is a director of a factory who is going to enlarge the assortment of production. Due to the constrained funds both for production and advertisement, only one of the three new alternative items can be produced. Each of these three options has advantages and disadvantages and thus the director asks three experts to evaluate these three options. However, the experts also noted that there are several important factors which cannot be either measured or foreseen properly at the moment of making decision about the new production. For example, the profit of the production is dependent with regard to the behaviour of other similar factories since they may offer a cheaper product at the same time. The costs of raw materials, which are different for each option, can change dramatically. In such situation, it is reasonable for the experts to describe their preferences in IFPRs which contain not only pure preferences but also hesitation connected with each option. Since each expert is independent and has no significant discrimination, the equal weights are assigned to them. The three IFPRs are as follows:

$$\tilde{R}^{(1)}=\begin{pmatrix}(0.5,0.5) & (0.1,0.9) & (0.5,0.4)\\ (0.9,0.1) & (0.5,0.5) & (0.5,0.3)\\ (0.4,0.5) & (0.3,0.5) & (0.5,0.5)\end{pmatrix}$$

$$\tilde{R}^{(2)}=\begin{pmatrix}(0.5,0.5) & (0.1,0.9) & (0.5,0.2)\\ (0.9,0.1) & (0.5,0.5) & (0.5,0.2)\\ (0.2,0.5) & (0.2,0.5) & (0.5,0.5)\end{pmatrix}$$

$$\tilde{R}^{(3)}=\begin{pmatrix}(0.5,0.5) & (0.2,0.8) & (0.1,0.2)\\ (0.8,0.2) & (0.5,0.5) & (0.6,0.3)\\ (0.2,0.1) & (0.3,0.6) & (0.5,0.5)\end{pmatrix}$$

In the following, we use the distinct consensus measures to calculate the consensus degree of this group respectively.

(1) The outranking flow based consensus measure

According to Eq. (8) and Eq. (9), the intuitionistic fuzzy positive outranking flow $\varphi_i^{(l)+}$ and the intuitionistic fuzzy negative outranking flow $\varphi_i^{(l)-}$ for alternative ξ_i ($i=1,2,3$) with respect to expert e_l ($l=1,2,3$) can be obtained, shown in Table 1.

Table 1. Outranking flow for each alternative with respect to different experts

	e_1	e_2	e_3
$(\varphi_1^{(l)+},\varphi_1^{(l)-})$	$((0.3292,0.6000),(0.7551,0.2236))$	$((0.3292,0.4243),(0.7172,0.2236))$	$((0.1515,0.4000),(0.6000,0.1414))$
$(\varphi_2^{(l)+},\varphi_2^{(l)-})$	$((0.7764,0.1732),(0.2063,0.6708))$	$((0.7764,0.1414),(0.1515,0.6708))$	$((0.7172,0.2449),(0.2517,0.6928))$
$(\varphi_3^{(l)+},\varphi_3^{(l)-})$	$((0.3519,0.5000),(0.5000,0.3464))$	$((0.2000,0.5000),(0.5000,0.2000))$	$((0.2517,0.6928),(0.7172,0.2449))$

Using Eq. (11), the distance between each pair of experts can be calculated, shown in Table 2.

Table 2. Distance between each pair of experts

	e_1	e_2	e_3
e_1	-	0.0997	0.1940*
e_2	0.0997	-	0.1889
e_3	0.1940	0.1889	-

Since the maximum distance is $d(e_1,e_3)=0.1940$, according to Definition 2, the outranking flow based consensus degree of this group is $CM_1=1-0.1940=0.8060$.

(2) The ordinal consensus measure

With this type of consensus measure, we should first obtain the ranking of the alternatives with respect to each individual IFPR and the aggregated IFPR. Using Model 1, the underlying priority vectors of the alternatives regarding to the individual IFPRs are

$$\begin{aligned}\tilde{\omega}^{(1)}&=(\tilde{\omega}_1^{(1)},\tilde{\omega}_2^{(1)},\tilde{\omega}_3^{(1)})^T\\&=((0.0909,0.9090),(0.8000,0.1636),(0.0727,0.8909))^T\end{aligned}\tag{29}$$

$$\begin{aligned}\tilde{\omega}^{(2)}&=(\tilde{\omega}_1^{(2)},\tilde{\omega}_2^{(2)},\tilde{\omega}_3^{(2)})^T\\&=((0.0909,0.9090),(0.7636,0.1273),(0.0364,0.8545))^T\end{aligned}\tag{30}$$

$$\begin{aligned}\tilde{\omega}^{(3)}&=(\tilde{\omega}_1^{(3)},\tilde{\omega}_2^{(3)},\tilde{\omega}_3^{(3)})^T\\&=((0.0409,0.8178),(0.4461,0.5539),(0.0818,0.4870))^T\end{aligned}\tag{31}$$

Using the SIFWG operator shown in Eq. (6), the overall weight vector of the alternatives is calculated:

$$\begin{aligned}\tilde{\omega}&=(\tilde{\omega}_1,\tilde{\omega}_2,\tilde{\omega}_3)^T\\&=((0.0697,0.8775),(0.6483,0.2260),(0.0600,0.7184))^T\end{aligned}$$

Calculating the value of the similarity function[36] $L(\tilde{\omega}_i)=\frac{1-\omega_i^v}{1+\omega_i^\pi}$ for each IFN $\tilde{\omega}_i$, we can yield the ranks of the alternatives with respect to each individual IFPR. In analogous, we can obtain the ranks from the overall IFPR as well. All the results are shown in Table 3.

Table 3. Ranking of the alternatives with respect different IFPRs

	$L(\tilde{\omega}_1^{(l)})$	$L(\tilde{\omega}_2^{(l)})$	$L(\tilde{\omega}_3^{(l)})$	Rankings
$\tilde{R}^{(1)}$	0.0910	0.8070	0.1053	$\xi_2 \succ \xi_3 \succ \xi_1$
$\tilde{R}^{(2)}$	0.0910	0.7869	0.1312	$\xi_2 \succ \xi_3 \succ \xi_1$
$\tilde{R}^{(3)}$	0.1596	0.4461	0.3584	$\xi_2 \succ \xi_3 \succ \xi_1$
$\tilde{R}$	0.1164	0.6876	0.2305	$\xi_2 \succ \xi_3 \succ \xi_1$

Since all the rankings derived from different IFPRs are the same, according to Eq. (14), the ordinal consensus degree for this group is $CM_2=1$.

(3) The α-cut based consensus measure

In Ref. 30, Szmidt and Kacprzyk used the α-cut based consensus measure to calculate the consensus degree of this group of the experts. At first, the IFPRs are divided into two kinds of preference relations:

$$U^{(1)}=\begin{bmatrix}0.5 & 0.1 & 0.5\\ 0.9 & 0.5 & 0.5\\ 0.4 & 0.3 & 0.5\end{bmatrix},\ \Delta^{(1)}=\begin{bmatrix}0.5 & 0 & 0.1\\ 0 & 0.5 & 0.2\\ 0.1 & 0.2 & 0.5\end{bmatrix}$$

$$U^{(2)}=\begin{bmatrix}0.5 & 0.1 & 0.5\\ 0.9 & 0.5 & 0.5\\ 0.2 & 0.2 & 0.5\end{bmatrix},\ \Delta^{(2)}=\begin{bmatrix}0.5 & 0 & 0.3\\ 0 & 0.5 & 0.3\\ 0.3 & 0.3 & 0.5\end{bmatrix}$$

$$U^{(3)}=\begin{bmatrix}0.5 & 0.2 & 0.1\\ 0.8 & 0.5 & 0.6\\ 0.2 & 0.3 & 0.5\end{bmatrix},\ \Delta^{(3)}=\begin{bmatrix}0.5 & 0 & 0.7\\ 0 & 0.5 & 0.1\\ 0.7 & 0.1 & 0.5\end{bmatrix}$$

For $U^{(l)}$ ($l=1,2,3$), set different value of α. According to Eq. (15), Eq. (16) and Eq. (17), it follows $K_{\min,0}=1$, $K_{\min,0.2}=0.63$, $K_{\min,0.4}=0.42$, $K_{\min,0.6}=0.33$, $K_{\min,0.8}=0.33$, and $K_{\min,1}=0$. By Eq. (19), we have $K_{\min}=0.44$. Similarly, for $\Delta^{(l)}$, ($l=1,2,3$), we can obtain the upper bound of the consensus degree is $K_{\max}=0.58$. Thus, the α-cut based consensus degree of this group is $CM_3=[0.44,0.58]$.

(4) The similarity based consensus measure

First calculate the similarity for each pair of experts concerning alternatives ξ_1, ξ_2 and ξ_3. With Eq. (20) and Eq. (21), it follows $Sim(e_1,e_2)_{12}=0.8$, $Sim(e_1,e_2)_{13}=0.1$, $Sim(e_1,e_2)_{23}=0.2$, thus, the average similarity for experts e_1 and e_2 is $Sim(e_1,e_2)=(0.8+0.1+0.2)/3=0.3667$.

In analogous, for experts e_1 and e_3, we have $Sim(e_1,e_3)_{12}=0.6$, $Sim(e_1,e_3)_{13}=0$, $Sim(e_1,e_3)_{23}=0.2$. Then, the average similarity for experts e_1 and e_3 is $Sim(e_1,e_3)=(0.6+0+0.2)/3=0.2667$. For experts e_2 and e_3, it follows $Sim(e_2,e_3)_{12}=0.6$, $Sim(e_2,e_3)_{13}=0$, $Sim(e_2,e_3)_{23}=0.15$. Therefore, the average similarity for experts e_2 and e_3 is $Sim(e_2,e_3)=(0.6+0+0.15)/3=0.25$.

Hence, the similarity based consensus degree of this group is $CM_4=(0.3666+0.266+0.25)/3=0.297$.

(5) The aggregation based consensus measure

In the following, we use the aggregation based consensus measure to calculate the consensus degree of this group. Since all the experts have equal weights, by Eq. (23), the overall IFPR of this group is

$$\tilde{R}=\begin{pmatrix}(0.5000,0.5000) & (0.1333,0.8667) & (0.3667,0.2667)\\ (0.8667,0.1333) & (0.5000,0.5000) & (0.5333,0.2667)\\ (0.2667,0.3667) & (0.2667,0.5333) & (0.5000,0.5000)\end{pmatrix}$$

Then, according to Eq. (1) and Eq. (25), we have

$$Sim(\tilde{r}_{11}^{(1)},\tilde{r}_{11})=Sim(\tilde{r}_{22}^{(1)},\tilde{r}_{22})=Sim(\tilde{r}_{33}^{(1)},\tilde{r}_{33})=0.5,$$
$$Sim(\tilde{r}_{12}^{(1)},\tilde{r}_{12})=Sim(\tilde{r}_{21}^{(1)},\tilde{r}_{21})=0.9584,$$
$$Sim(\tilde{r}_{23}^{(1)},\tilde{r}_{23})=Sim(\tilde{r}_{32}^{(1)},\tilde{r}_{32})=0.5,$$
$$Sim(\tilde{r}_{13}^{(1)},\tilde{r}_{13})=Sim(\tilde{r}_{31}^{(1)},\tilde{r}_{31})=0.8751;$$
$$Sim(\tilde{r}_{11}^{(2)},\tilde{r}_{11})=Sim(\tilde{r}_{22}^{(2)},\tilde{r}_{22})=Sim(\tilde{r}_{33}^{(2)},\tilde{r}_{33})=0.5,$$
$$Sim(\tilde{r}_{12}^{(2)},\tilde{r}_{12})=Sim(\tilde{r}_{21}^{(2)},\tilde{r}_{21})=0.9584,$$
$$Sim(\tilde{r}_{23}^{(2)},\tilde{r}_{23})=Sim(\tilde{r}_{32}^{(2)},\tilde{r}_{32})=0.6364,$$
$$Sim(\tilde{r}_{13}^{(2)},\tilde{r}_{13})=Sim(\tilde{r}_{31}^{(2)},\tilde{r}_{31})=0.7692;$$
$$Sim(\tilde{r}_{11}^{(3)},\tilde{r}_{11})=Sim(\tilde{r}_{22}^{(3)},\tilde{r}_{22})=Sim(\tilde{r}_{33}^{(3)},\tilde{r}_{33})=0.5,$$
$$Sim(\tilde{r}_{12}^{(3)},\tilde{r}_{12})=Sim(\tilde{r}_{21}^{(3)},\tilde{r}_{21})=0.9091,$$
$$Sim(\tilde{r}_{23}^{(3)},\tilde{r}_{23})=Sim(\tilde{r}_{32}^{(3)},\tilde{r}_{32})=0.5,$$
$$Sim(\tilde{r}_{13}^{(3)},\tilde{r}_{13})=Sim(\tilde{r}_{31}^{(3)},\tilde{r}_{31})=0.7692.$$

Thus, from Eq. (24), it follows

$$Sim(\tilde{R}^{(1)},\tilde{R})=0.6852,\ Sim(\tilde{R}^{(2)},\tilde{R})=0.6920,$$
$$Sim(\tilde{R}^{(3)},\tilde{R})=0.6507$$

Therefore, by Eq. (26), the aggregation based consensus degree of this group is $CM_5=(0.6852+0.6920+0.6507)/3=0.6760$.

(6) The distance based consensus measure

If using the distance based consensus measure to calculate the consensus degree, the first thing we should do is to derive the underlying weight vector from each individual IFPR. Based on Model 1, the weight vector can be obtained, shown as Eq. (29)-Eq. (31). Then, according to Eq. (27), the distance between each pair of experts can be calculated (shown in Table 4).

Table 4. Pairwise distances of the experts

	e_1	e_2	e_3
e_1	-	0.0485	0.3118
e_2	0.0485	-	0.3118
e_3	0.3118	0.3118	-

From Table 4, we can find that the maximum distance between each pair of experts is 0.3118 . According to Eq. (28), the distance based consensus degree for this group is $CM_6 = 1 - 0.3118 = 0.6882$. ■

From the above example, we can find that with different consensus measures, different consensus degrees will be obtained. Thus, how to choose the most appropriate consensus measure is still an open question since its answer may be depended on the specific application problems being addressed. However, the above mentioned example at least shows that the proposed consensus measures outperform the four existing ones in this particular application example. As it is shown in Table 3, the ranking orders among the three experts are the same, which indicates the very high consensus with the group. For such a case with the high consensus, $CM_2 = 1$ and CM_1 =0.806 fit this circumstance best and the second best respectively in comparing with the other four measures. Further, this illustrated example shows that CM_1 is the easiest consensus measure as it only needs to calculate the outranking flows and the distances between the flows. The distance based consensus measure CM_6 is a little more complicated than CM_1 because for this type of consensus measure, the underlying priority vector should be derived first. Meanwhile, the ordinal consensus measure CM_2 is slightly simple than CM_6 owing to the fact that there is no need to compute the distance between each pair of priorities for CM_2. Both CM_4 and CM_5 are based on the similarity measures, but for CM_5, we should first aggregate the individual IFPRs into an overall IFPR, thus it is slightly more complicated than CM_4. The α-cut based consensus measure CM_3 is the most complicated consensus measure among these six type of consensus measures, because with such consensus measure, we should derive the consensus degree for two preference relations.

Generally, in practical applications, if a decision maker does not want to know the divergence between the experts' preferences in details, he/she does not need to obtain the crisp value of the consensus degree for a group. In other words, experts in a group do not have to agree with each other absolutely in order to reach a consensus. For example, in judging figure skating, there is no expectation that all experts will eventually converge to an agreement. On the contrary, it is common that these judges produce diversified preferences. In such a case, the analyst always eliminates the high and low extreme opinions and averages the rest. From this point of view, the second consensus measure, CM_2, is most appropriate to measure the consensus degree. Meanwhile, as we can see from Table 3, although the underlying priorities are quite different, the rankings from different IFPRs are the same, which further makes the ordinal consensus degree CM_2 of the group be equal to zero. That is to say, the ordinal consensus degree CM_2 is somehow robust.

In the following, we would give a specific procedure for GDM with intuitionistic fuzzy preference information based on this ordinal consensus measure.

5. A Procedure for Intuitionistic Fuzzy GDM Based on the Ordinal Consensus Measure

In order to aid a decision maker to find the final solution for an intuitionistic fuzzy GDM problem, based on the ordinal consensus measure, we give a procedure for practical GDM with IFPRs.

Algorithm 2

Step 1: The decision maker determines the weight vector of a group of experts $E = \{e_1, e_2, \cdots, e_l, \cdots, e_s\}$ as $\lambda = (\lambda_1, \lambda_2, ..., \lambda_s)^T$ with $\lambda_l > 0$, ($l = 1,2,...,s$) and $\sum_{l=1}^{s} \lambda_l = 1$. Each expert e_l expresses his/her opinions by the IFPR $\tilde{R}^{(l)} = \left(\tilde{r}_{ij}^{(l)}\right)_{n \times n}$ in the form of Eq. (2). Go to the next step.

Step 2: For each individual IFPR $\tilde{R}^{(p)(l)} = \left(\tilde{r}_{ij}^{(p)(l)}\right)_{n\times n}$, the decision maker checks the consistency of each IFPR via Eq. (4). For those IFPRs with unacceptable consistency, the decision maker returns them to the corresponding experts for re-evaluation. If all the IFPRs are with acceptable consistency, then go to the next step.

Step 3: Determine the ordinal consensus measure for each expert by Eq. (14). Go to the next step.

Step 4: If all consensus measures for the group of experts reach to 1, then the ranking of the alternatives is obtained and the procedure ends; otherwise, exclude the top 2 experts whose consensus degrees are the first and smallest ones in the group, then go to the next step.

Step 5: Use the SIFWG operator shown in Eq. (6) to aggregate the weight vectors derived from the IFPRs given by the rest members of the group, and then rank the alternatives according to the obtained overall weighting vector.

Step 6: End.

It is noted that excluding the highest and the lowest scores is the common way in many practical evaluation process. However, excluding the high and low extreme opinions only takes place in the case that there are many different candidate alternatives. If there are only three experts, it is easy to reach group consensus.

6. Conclusions

GDM with IFPRs is an important research topic in many fields such as financial, managerial, engineering and medical domains. It is also a very complicated problem owing to the conflicting opinions from experts. To depict the degree of agreement among a group of experts, consensus measures are needed to be developed and they are significantly important in GDM with intuitionistic fuzzy preference information. In this paper, we have proposed two novel consensus measures, which are the outranking flow based consensus measure and the ordinal consensus measure, to help the analyst to describe and assess the degree of agreement among the experts in a group. Moreover, we have reviewed all the other existing consensus measures for intuitionistic fuzzy GDM in the literature, such as the α -cut based consensus measure, the similarity based consensus measure, the aggregation based consensus measure, and the distance based consensus measure. We have compared these distinct consensus measures to show the different between these consensus measures and highlight the advantages of the proposed consensus measures. Finally, in order to help a decision maker to find a final solution for an intuitionistic fuzzy GDM problem, an intuitionistic fuzzy GDM procedure has been proposed based on the ordinal consensus measure.

In the future, we will apply the intuitionistic fuzzy GDM procedure to solve some practical GDM problems. Meanwhile, the multi-person multi-criteria GDM problem within the context of intuitionist fuzzy circumstance is also a good question to be investigated.

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