

Stochastic Non-autonomous Lotka-Volterra Mutualism systems with Impulse Jump and Markov Switching

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Abstract. In this paper, we investigate a stochastic non-autonomous Lotka-Volterra mutualism with impulse jump and Markov switching, and two sufficient conditions for stochastic persistence are established

Keywords: Brownian motion; Impulse jump; Markov switching; Generalized It formul; Stochastic persistence.

1. Introduction

The classical non-autonomous Lotka-Volterra mutualism system can be described as follows:

$$\frac{dx_i(t)}{dt} = x_i(t) \left[r_i(t) - a_{ii}(t)x_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(t)x_j(t) \right] \quad (1)$$

Extensive literature concerned mutualism system (1) and its generalized forms.

The above mentioned papers are all deterministic models, which do not consider the effect of environmental noise for population system. In fact, population system is often subject to effect by environmental noise (see [1-3]). In paper [2-3], some result on the nonexplosion, boundedness and persistence for stochastic population systems have been developed. Particularly, Ji and Jiang in [3] studied nonautonomous two-species stochastic Lotka-Volterra mutualism model

$$\begin{cases} dx_1(t) = x_1(t) \left[(r_1(t) - a_{11}(t)x_1(t) + a_{12}(t)x_2(t))dt + \sigma_1(t)dB_1(t) \right] \\ dx_2(t) = x_2(t) \left[(r_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t))dt + \sigma_2(t)dB_2(t) \right] \end{cases} \quad (2)$$

For the systems (2) and its generalized forms, many interesting results have been obtained.

On the other hand, the growth of species usually undergoes some discrete changes of relatively short time interval at some fixed times, such as drought, flooding, earthquake, planting, harvesting etc. From point view of mathematic, the sudden changes could be described by impulses. In this case, impulsive effects should be taken into account system (2). Some results (see [4-7]) have been proposed for stochastic systems with impulsive effects. Up to now, little research has been done about stochastic non-autonomous Lotka-Volterra mutualism with both impulse effects and Markov switching. Motivated by these, in this paper we consider the following non-autonomous two-species stochastic Lotka-Volterra mutualism with impulse jump and Markov switching system:

$$\begin{cases} dx_1(t) = x_1(t) \left[r_1(\xi(t)) - a_{11}(\xi(t))x_1(t) + a_{12}(\xi(t))x_2(t) \right] dt \\ \quad + \sigma_1(\xi(t))x_1(t)dB_1(t), \quad t \neq t_k, k \in N \\ dx_2(t) = x_2(t) \left[r_2(\xi(t)) - a_{21}(\xi(t))x_1(t) - a_{22}(\xi(t))x_2(t) \right] dt \\ \quad + \sigma_2(\xi(t))x_2(t)dB_2(t), \quad t \neq t_k, k \in N \\ x_1(t_k^+) - x_1(t_k) = b_{1k}x_1(t_k), k \in N \\ x_2(t_k^+) - x_2(t_k) = b_{2k}x_2(t_k), k \in N \end{cases} \quad (3)$$

Where N denotes the set of positive integers, $0 < t_1 < t_2 \cdots, \lim_{k \rightarrow +\infty} t_k = +\infty$. On count of biological meanings we impose the additional restrictions on systems (3),

$$b_{ik} > -1, i = 1, 2, \dots, n, k \in N.$$

When $b_{ik} > 0$, the impulsive effects denote planting, while $b_{ik} < 0$ represent harvesting [14]. $\xi(t)$ is Markov switching take values in $S = \{1, 2, \dots, N\}$, switching from one mode to the others according to movement of the Markov chain $\xi(t)$. $B_i(t), i = 1, 2$ are mutually independent one dimensional standard Brownian motions with $B_i(0) = 0$.

2. Stochastic persistence

Let R_+^2 denote positive cone of R^2 . For $x \in R^2, |x| = |x_1| + |x_2|$. Moreover, we assume that $\inf_{t \geq 0} a_{ii}(\xi(t)) > 0$ for all $i = 1, 2$. and $a_{ij}(\xi(t)) > 0$ for all $i = 1, 2$ with $j \neq i$.

Studying a population system, we pay more attention on whether the system is persistent. In this section, we first show that the solution is stochastic permanence. Before give the main theorems, we first give some famous concepts, lemmas and assumptions (see e.g. [7]).

Definition 1 system (3) is said to be stochastically permanent if for every $\varepsilon \in (0, 1)$, there are constants $M = M(\varepsilon), N = N(\varepsilon)$ such that

$$\liminf_{t \rightarrow +\infty} \{x(t) \geq M\} \geq 1 - \varepsilon, \quad \liminf_{t \rightarrow +\infty} \{x(t) \leq N\} \geq 1 - \varepsilon.$$

Assumption 1 For some $u \in S, q_{iu} > 0 (\forall i \neq u)$.

Assumption 2 $\sum_{k=1}^n \pi_u \hat{\beta}(u) > 0, \hat{\beta}(u) = \check{b}(u) - \frac{1}{2} \sigma^2(u), \check{b}(u) = \min_{1 \leq i, j \leq 2} b_{ij}(u), u \in S.$

Assumption 3 $\hat{\beta}(u) > 0, u \in S.$

Lemma 1 (see e.g. [4]). Assumptions 1 and 2 imply that there exists a constant $\theta > 0$ such that the matrix $A(\theta)$ is a nonsingular M-matrix, where

$$A(\theta) = \text{diag}(\xi_1(\theta), \xi_2(\theta), \dots, \xi_n(\theta)) - \Gamma, \quad \xi_u(\theta) = \theta \hat{\beta}(u) - \frac{1}{2} \theta^2 \sigma^2(u), \quad \forall u \in S.$$

Lemma 2 Assumption 3 imply that there exists a constant $\theta > 0$ such that the matrix $A(\theta)$ is a nonsingular M-matrix.

Assumption 4 There exist two constants $m > 0$ and $M > 0$ such that $m \leq \prod_{0 < t_k < t} (1 + b_{ik}) y_i(t) \leq M$ for all $t > 0, i = 1, 2$.

Theorem 2 If assumption (1), (2) and (4) hold, then system (3) is stochastically permanent.

Proof. Firstly, let us show that for given $\varepsilon \in (0, 1)$, there exists a positive constant M such that $\liminf_{t \rightarrow +\infty} \{x(t) \geq M\} \geq 1 - \varepsilon$. Applying the generalized Itô formula, we compute

$$\begin{aligned} d\left(\frac{1}{y_1}\right) &= -\frac{1}{y_1^2} dy_1 + \frac{1}{y_1^3} (d y_1 \cdot dy_1) \\ &= -\frac{1}{y_1} \left[r_1(\xi) - a_{11}(\xi) \prod_{0 < t_k < t} (1 + b_{1k}) y_1 + a_{12}(\xi) \prod_{0 < t_k < t} (1 + b_{2k}) y_2 \right] dt \\ &\quad + \frac{1}{y_1} \sigma_1^2(\xi) y_1 dt - \frac{1}{y_1} \sigma_1(\xi) dB_1(t). \end{aligned}$$

$$d\left(\frac{1}{y_2}\right) = -\frac{1}{y_2^2} dy_2 + \frac{1}{y_2^3} (d y_2 \cdot dy_2)$$

$$= -\frac{1}{y_2} \left[r_2(\xi) + a_{21}(\xi) \prod_{0 < t_k < t} (1 + b_{1k}) y_1 - a_{22}(\xi) \prod_{0 < t_k < t} (1 + b_{2k}) y_2 \right] dt + \frac{1}{y_2} \sigma_2^2(\xi) dt - \frac{1}{y_2} \sigma_2(\xi) dB_2(t).$$

For θ given in lemma 3, by lemma 2, there is a vector $\vec{q} = (q_1, q_2, \dots, q_N)^T \geq 0$ such that $A(\theta)\vec{q} \leq 0$, namely

$$q_u \left(\theta \hat{\beta}(u) - \frac{1}{2} \theta^2 \check{\sigma}^2(u) \right) - \sum_{l=1}^N \gamma_{ul} q_l \text{ for all } 1 \leq u \leq N.$$

Define $V_1(y, u) = q_u (1 + y_1^{-1})^\theta + q_u (1 + y_2^{-1})^\theta$. Then, by the generalized Itô formula, we have

$$EV_1(y(t), \xi(t)) = V_1(y(0), \xi(0)) + E \int_0^t LV_1(y(s), \xi(s)) ds,$$

Where

$$\begin{aligned} LV_1(y, u) &= q_u \theta \left(1 + \frac{1}{y_1}\right)^{\theta-1} d \frac{1}{y_1} + \frac{1}{2} q_u \theta (\theta - 1) \left(1 + \frac{1}{y_1}\right)^{\theta-2} d \left(\frac{1}{y_1}\right)^2 \\ &\quad + q_u \theta \left(1 + \frac{1}{y_2}\right)^{\theta-1} d \frac{1}{y_2} + \frac{1}{2} q_u \theta (\theta - 1) \left(1 + \frac{1}{y_2}\right)^{\theta-2} d \left(\frac{1}{y_2}\right)^2 + \sum_{l=1}^N \gamma_{ul} V_1(y, l) q_l \\ &= q_u \theta \left(1 + \frac{1}{y_1}\right)^{\theta-2} \left\{ \left(1 + \frac{1}{y_1}\right) \left[-\frac{1}{y_1} (r_1(\xi) - a_{11}(\xi) \prod_{0 < t_k < t} (1 + b_{1k}) y_1 + a_{12}(\xi) \prod_{0 < t_k < t} (1 + b_{2k}) y_2) \right. \right. \\ &\quad \left. \left. + \frac{1}{y_1} \sigma_1^2(\xi) \right] + \frac{1}{2} (\theta - 1) \left(\frac{\sigma_1(\xi)}{y_1}\right)^2 \right\} + q_u \theta \left(1 + \frac{1}{y_2}\right)^{\theta-2} \left\{ \left(1 + \frac{1}{y_2}\right) \left[-\frac{1}{y_2} (r_2(\xi) + a_{21}(\xi) \prod_{0 < t_k < t} (1 + b_{1k}) y_1 \right. \right. \right. \\ &\quad \left. \left. - a_{22}(\xi) \prod_{0 < t_k < t} (1 + b_{2k}) y_2) + \frac{1}{y_2} \sigma_2^2(\xi) \right] + \frac{1}{2} (\theta - 1) \left(\frac{\sigma_2(\xi)}{y_2}\right)^2 \right\} + \left(1 + \frac{1}{y_1}\right)^\theta \sum_{l=1}^N \gamma_{ul} q_l + \left(1 + \frac{1}{y_2}\right)^\theta \sum_{l=1}^N \gamma_{ul} q_l \\ &= \left(1 + \frac{1}{y_1}\right)^{\theta-2} \left\{ -\frac{1}{y_1} q_u \theta [r_1(\xi) - \sigma_1^2(\xi)] - \frac{1}{2} (\theta - 1) \sigma_1^2(\xi) \right\} + \frac{1}{y_1} q_u \theta [-r_1(\xi) + \sigma_1^2(\xi) + a_{11}(\xi) \prod_{0 < t_k < t} (1 + b_{1k})] \\ &\quad - \frac{y_2}{y_1} \left[q_u \theta (a_{12}(\xi) \prod_{0 < t_k < t} (1 + b_{2k}) + \frac{a_{12}(\xi) \prod_{0 < t_k < t} (1 + b_{2k})}{y_1}) \right] + a_{11}(\xi) \prod_{0 < t_k < t} (1 + b_{1k}) + \left(1 + \frac{1}{y_1}\right)^\theta \sum_{l=1}^N \gamma_{ul} q_l \\ &\quad + \left(1 + \frac{1}{y_2}\right)^{\theta-2} \left\{ -\frac{1}{y_2} q_u \theta [r_2(\xi) - \sigma_2^2(\xi)] - \frac{1}{2} (\theta - 1) \sigma_2^2(\xi) \right\} + \frac{1}{y_2} q_u \theta [-r_2(\xi) + \sigma_2^2(\xi) \\ &\quad + a_{21}(\xi) \prod_{0 < t_k < t} (1 + b_{1k})] \\ &\quad - \frac{y_1}{y_2} \left[q_u \theta (a_{21}(\xi) \prod_{0 < t_k < t} (1 + b_{1k}) + \frac{a_{21}(\xi) \prod_{0 < t_k < t} (1 + b_{1k})}{y_2}) \right] \\ &\quad + a_{22}(\xi) \prod_{0 < t_k < t} (1 + b_{2k}) + \left(1 + \frac{1}{y_2}\right)^\theta \sum_{l=1}^N \gamma_{ul} q_l \\ &= \left(1 + \frac{1}{y_1}\right)^{\theta-2} \left\{ -\frac{1}{y_1} [q_u (\theta r_1(u) - \frac{1}{2} \theta \sigma_1^2(u) - \frac{1}{2} \theta^2 \sigma_1^2(u)) - \sum_{l=1}^N \gamma_{ul} q_l] + \frac{1}{y_1} [q_u \theta (a_{11}(u) \prod_{0 < t_k < t} (1 + b_{1k}) \right. \right. \\ &\quad \left. \left. - r_1(u) + \sigma_1^2(u)) + 2 \sum_{l=1}^N \gamma_{ul} q_l] - \frac{y_2}{y_1} [q_u \theta (a_{12}(u) \prod_{0 < t_k < t} (1 + b_{2k}) + \frac{a_{12}(u) \prod_{0 < t_k < t} (1 + b_{2k})}{y_1}) \right] \right\} \\ &\quad + [q_u \theta a_{11}(u) \prod_{0 < t_k < t} (1 + b_{1k}) + \sum_{l=1}^N \gamma_{ul} q_l] + \left(1 + \frac{1}{y_2}\right)^{\theta-2} \left\{ -\frac{1}{y_2} [q_u (\theta r_2(u) - \frac{1}{2} \theta \sigma_2^2(u) - \frac{1}{2} \theta^2 \sigma_2^2(u)) \right. \\ &\quad \left. - \sum_{l=1}^N \gamma_{ul} q_l] + \frac{1}{y_2} [q_u \theta (a_{22}(u) \prod_{0 < t_k < t} (1 + b_{2k}) - r_2(u) + \sigma_2^2(u)) + 2 \sum_{l=1}^N \gamma_{ul} q_l] - \frac{y_1}{y_2} [q_u \theta (a_{21}(u) \prod_{0 < t_k < t} (1 + b_{1k}) \right. \right. \end{aligned}$$

$$a_{21}(u) \prod_{0 < t_k < t} (1 + b_{1k}) + \frac{1}{y_2} \left. \right] + [q_u \theta a_{22}(u) \prod_{0 < t_k < t} (1 + b_{2k}) + \sum_{l=1}^N \gamma_{ul} q_l \}.$$

Now, choose $\eta > 0$ sufficiently small such that it satisfies

$$q_u (\theta r_i(u) - \frac{1}{2} \theta^2 \sigma_i(u)) - \sum_{l=1}^N \gamma_{ul} q_l - \eta q_u > 0 \text{ for all } 1 \leq u \leq N, i = 1, 2.$$

Define $V_2(y, u) = e^{\eta t} V_1(y, u) = e^{\eta t} q_u (1 + y_1^{-1})^\theta + e^{\eta t} q_u (1 + y_2^{-1})^\theta$. Then, by the generalized Itô formula

$$EV_2(y(t), \xi(t)) = V_2(y(0), \xi(0)) + E \int_0^t LV_2(y(s), \xi(s)) ds,$$

Where

$$LV_2(y, u) = \eta e^{\eta t} V_1(y, u) + e^{\eta t} LV_1(y, u)$$

$$\begin{aligned} &= e^{\eta t} (1 + y_1^{-1})^{\theta-2} \left\{ -\frac{1}{y_1^2} [q_u \theta r_1(u) - \frac{1}{2} q_u \theta \sigma_1^2(u) - \sum_{l=1}^N \gamma_{ul} q_l - \eta q_u] + \frac{1}{y_1} [q_u \theta a_{11}(u) \prod_{0 < t_k < t} (1 + b_{1k}) \right. \\ &\quad - q_u \theta r_1(u) + q_u \theta \sigma_1^2(u) + 2 \sum_{l=1}^N \gamma_{ul} q_l + 2 \eta q_u] - \frac{y_2}{y_1} [q_u \theta a_{12}(u) \prod_{0 < t_k < t} (1 + b_{2k}) + \frac{q_u \theta a_{12}(u) \prod_{0 < t_k < t} (1 + b_{2k})}{y_1} \\ &\quad \left. + [q_u \theta a_{11}(u) \prod_{0 < t_k < t} (1 + b_{1k}) + \sum_{l=1}^N \gamma_{ul} q_l + \eta q_u] + e^{\eta t} (1 + y_2^{-1})^{\theta-2} \left\{ -\frac{1}{y_2^2} [q_u \theta r_2(u) - \frac{1}{2} q_u \theta \sigma_2^2(u) - \sum_{l=1}^N \gamma_{ul} q_l - \eta q_u] \right. \right. \\ &\quad \left. + \frac{1}{y_2} [q_u \theta a_{22}(u) \prod_{0 < t_k < t} (1 + b_{2k}) - q_u \theta r_2(u) + q_u \theta \sigma_2^2(u) + 2 \sum_{l=1}^N \gamma_{ul} q_l + 2 \eta q_u] - \frac{y_1}{y_2} [q_u \theta a_{21}(u) \prod_{0 < t_k < t} (1 + b_{1k}) \right. \\ &\quad \left. \left. + \frac{q_u \theta a_{21}(u) \prod_{0 < t_k < t} (1 + b_{1k})}{y_2} \right] + [q_u \theta a_{22}(u) \prod_{0 < t_k < t} (1 + b_{2k}) + \sum_{l=1}^N \gamma_{ul} q_l + \eta q_u] \right\} \\ &\leq e^{\eta t} (1 + y_1^{-1})^{\theta-2} \left\{ -\frac{1}{y_1^2} [q_u \theta r_1(u) - \frac{1}{2} q_u \theta \sigma_1^2(u) - \sum_{l=1}^N \gamma_{ul} q_l - \eta q_u] + \frac{1}{y_1} [q_u \theta a_{11}(u) M \right. \\ &\quad - q_u \theta r_1(u) + q_u \theta \sigma_1^2(u) + 2 \sum_{l=1}^N \gamma_{ul} q_l + 2 \eta q_u] + [q_u \theta a_{11}(u) M + \sum_{l=1}^N \gamma_{ul} q_l + \eta q_u] \left. \right\} \\ &\quad + e^{\eta t} (1 + y_2^{-1})^{\theta-2} \left\{ -\frac{1}{y_2^2} [q_u \theta r_2(u) - \frac{1}{2} q_u \theta \sigma_2^2(u) - \sum_{l=1}^N \gamma_{ul} q_l - \eta q_u] + \frac{1}{y_2} [q_u \theta a_{22}(u) M \right. \\ &\quad \left. - q_u \theta r_2(u) + q_u \theta \sigma_2^2(u) + 2 \sum_{l=1}^N \gamma_{ul} q_l + 2 \eta q_u] + [q_u \theta a_{22}(u) M + \sum_{l=1}^N \gamma_{ul} q_l + \eta q_u] \right\} \\ &=: e^{\eta t} H(y, u), \end{aligned}$$

where

$$\begin{aligned} H(y, u) &= (1 + y_1^{-1})^{\theta-2} \left\{ -\frac{1}{y_1^2} [q_u \theta r_1(u) - \frac{1}{2} q_u \theta \sigma_1^2(u) - \sum_{l=1}^N \gamma_{ul} q_l - \eta q_u] + \frac{1}{y_1} [q_u \theta a_{11}(u) M \right. \\ &\quad - q_u \theta r_1(u) + q_u \theta \sigma_1^2(u) + 2 \sum_{l=1}^N \gamma_{ul} q_l + 2 \eta q_u] + [q_u \theta a_{11}(u) M + \sum_{l=1}^N \gamma_{ul} q_l + \eta q_u] \left. \right\} \\ &\quad + (1 + y_2^{-1})^{\theta-2} \left\{ -\frac{1}{y_2^2} [q_u \theta r_2(u) - \frac{1}{2} q_u \theta \sigma_2^2(u) - \sum_{l=1}^N \gamma_{ul} q_l - \eta q_u] + \frac{1}{y_2} [q_u \theta a_{22}(u) M \right. \\ &\quad \left. - q_u \theta r_2(u) + q_u \theta \sigma_2^2(u) + 2 \sum_{l=1}^N \gamma_{ul} q_l + 2 \eta q_u] + [q_u \theta a_{22}(u) M + \sum_{l=1}^N \gamma_{ul} q_l + \eta q_u] \right\}. \end{aligned}$$

By the definition of η , $H(y, u)$ is upper bounded in $R_+ \times S$, This implies

$$\sqrt[q]{Ee^{m^t}(1+y_1^{-1})^\theta + e^{m^t}(1+y_2^{-1})^\theta} \leq \hat{q}(1+y_1^{-1}(0))^\theta + \hat{q}(1+y_2^{-1}(0))^\theta + \frac{H_1(e^{m^t}-1)}{\eta}.$$

That is to say

$$\limsup_{t \rightarrow +\infty} E[y_1^{-\theta}(t)] \leq \limsup_{t \rightarrow +\infty} E[(1+y_1^{-1}(t))^\theta + (1+y_2^{-1}(t))^\theta] \leq \frac{H_1}{\sqrt[q]} =: H_2.$$

Consequently

$$\limsup_{t \rightarrow +\infty} E[x_1^{-\theta}(t)] \leq \limsup_{t \rightarrow +\infty} E\left[\prod_{0 < t_k < t} (1+b_{1k})y_1\right]^{-\theta} \leq m^{-\theta}H_2 =: H_3$$

$$\limsup_{t \rightarrow +\infty} E[x_2^{-\theta}(t)] \leq \limsup_{t \rightarrow +\infty} E\left[\prod_{0 < t_k < t} (1+b_{2k})y_2\right]^{-\theta} \leq H_3.$$

For any given $\varepsilon > 0$, let $M = \left(\frac{\varepsilon}{H_2}\right)^{\frac{1}{\theta}}$, by the Chebyshev inequality, we can obtain that

$$P\{x_i(t) < M\} = P\{x_i^{-\theta}(t) < M^{-\theta}\} \leq M^\theta E[x_i^{-\theta}(t)]; \quad i=1,2.$$

Then,

$$\limsup_{t \rightarrow +\infty} P\{x_i(t) < M\} \leq MH_2 = \varepsilon.$$

Consequently

$$\liminf_{t \rightarrow +\infty} P\{x_i(t) \geq M\} \geq 1 - \varepsilon, \quad i=1,2.$$

Next we prove that for any given $\varepsilon > 0$, there exists a positive constant N such that $\liminf_{t \rightarrow +\infty} P\{x_i(t) \leq N\} \geq 1 - \varepsilon, \quad i=1,2.$

If $a_{11}(\xi)a_{22}(\xi) - a_{12}(\xi)a_{21}(\xi) > 0, \xi \in S$, then there exist positive numbers $c_1(\xi)$ and $c_2(\xi)$ for each $\xi \in S$ such that

$$-\lambda := \max_{\xi \in S} \{\lambda_{\max}^+(C(\xi)A(\xi) + A^T(\xi)C(\xi))\} < 0 \tag{4}$$

$$\text{Where } A(\xi) = \begin{pmatrix} -a_{11}(\xi) \prod_{0 < t_k < t} (1+b_{1k}) & a_{12}(\xi) \prod_{0 < t_k < t} (1+b_{2k}) \\ a_{21}(\xi) \prod_{0 < t_k < t} (1+b_{1k}) & -a_{22}(\xi) \prod_{0 < t_k < t} (1+b_{2k}) \end{pmatrix}, \quad C(\xi) = \begin{pmatrix} c_1(\xi) & 0 \\ 0 & c_2(\xi) \end{pmatrix}.$$

Define $V(y) = h_1y_1 + h_2y_2$ for $y \in \mathbb{R}_+^2$, where $h_1 > 0$ and $h_2 > 0$. Applying the generalized Itô formula yields

$$dV(y) = LV(y) + h_1\sigma_1(\xi)y_1dB_1(t) + h_2\sigma_2(\xi)y_2dB_2(t)$$

Where

$$\begin{aligned} LV(y) &= h_1y_1[r_1(\xi) - a_{11}(\xi) \prod_{0 < t_k < t} (1+b_{1k})y_1 + a_{12}(\xi) \prod_{0 < t_k < t} (1+b_{2k})y_2] \\ &\quad + h_2y_2[r_2(\xi) + a_{21}(\xi) \prod_{0 < t_k < t} (1+b_{1k})y_1 - a_{22}(\xi) \prod_{0 < t_k < t} (1+b_{2k})y_2] \\ &= h_1r_1(\xi)y_1 + h_2r_2(\xi)y_2 + \frac{1}{2}y^T(C(\xi)A + A^TC(\xi))y \\ &\leq h_1r_1(\xi)y_1 + h_2r_2(\xi)y_2 - \frac{\lambda}{2}(y_1^2 + y_2^2) \end{aligned}$$

(4) implies that there exists a constant $H_4 > 0$ such that $LV(y) < H_4$. Making use of the generalized Itô formula again leads to

$$\begin{aligned}
 d[e^t V(y)] &= e^t (h_1 y_1 + h_2 y_2) dt + e^t (LV(y)) dt + e^t [h_1 \sigma_1(\xi) y_1 dB_1(t) + h_2 \sigma_2(\xi) y_2 dB_2(t)] \\
 &\leq e^t [h_1 y_1 + h_2 y_2 + h_1 r_1(\xi) y_1 + h_2 r_2(\xi) y_2 - \frac{\lambda}{2} (y_1^2 + y_2^2)] dt + e^t [h_1 \sigma_1(\xi) y_1 dB_1(t) + h_2 \sigma_2(\xi) y_2 dB_2(t)] \\
 &\leq H_4 e^t dt + e^t [h_1 \sigma_1(\xi) y_1 dB_1(t) + h_2 \sigma_2(\xi) y_2 dB_2(t)]. \\
 d[e^t V(y)] &= e^t (h_1 y_1 + h_2 y_2) dt + e^t (LV(y)) dt + e^t [h_1 \sigma_1(\xi) y_1 dB_1(t) + h_2 \sigma_2(\xi) y_2 dB_2(t)] \\
 &\leq e^t [h_1 y_1 + h_2 y_2 + h_1 r_1(\xi) y_1 + h_2 r_2(\xi) y_2 - \frac{\lambda}{2} (y_1^2 + y_2^2)] dt + e^t [h_1 \sigma_1(\xi) y_1 dB_1(t) + h_2 \sigma_2(\xi) y_2 dB_2(t)] \\
 &\leq H_4 e^t dt + e^t [h_1 \sigma_1(\xi) y_1 dB_1(t) + h_2 \sigma_2(\xi) y_2 dB_2(t)].
 \end{aligned}$$

Integrating from 0 to t and taking expectations on both, we get

$$e^t EV(y(t)) = V(y(0)) + H_4(e^t - 1) = h_1 y_{10} + h_2 y_{20} + H_4(e^t - 1).$$

That is to say, $\limsup_{t \rightarrow +\infty} E[V(y(t))] \leq H_4$. On the other hand, it is easy to see

that $y_1 + y_2 \leq \frac{V(y(t))}{\min\{h_1, h_2\}}$. Therefore,

$$\begin{aligned}
 \limsup_{t \rightarrow +\infty} E[x_1(t)] &\leq \limsup_{t \rightarrow +\infty} E\left[\prod_{0 < t_k < t} (1 + b_{1k}) y_1(t)\right] \leq M \limsup_{t \rightarrow +\infty} E y_1(t) \leq M \limsup_{t \rightarrow +\infty} \frac{V(y(t))}{\min\{h_1, h_2\}} \\
 &\leq \frac{MH_4}{\min\{h_1, h_2\}} =: H_5, \text{ i.e. } \limsup_{t \rightarrow +\infty} E[x_1(t)] \leq H_5.
 \end{aligned}$$

Thus for any given $\varepsilon > 0$, Then by virtue of

Chebyshev inequality, we can derive that

$$P\{x_i(t) > N\} \leq \frac{E[x_i(t)]}{N} = \frac{\varepsilon}{H_5} E[x_i(t)], \quad i = 1, 2.$$

That is to say

$$\limsup_{t \rightarrow +\infty} P\{x_i(t) > N\} \leq \limsup_{t \rightarrow +\infty} \frac{E[x_i(t)]}{N} = \limsup_{t \rightarrow +\infty} \frac{\varepsilon}{H_5} E[x_i(t)] \leq \varepsilon, \quad i = 1, 2.$$

Theorem 2 If assumption (3) and (4) hold, then system (3) is stochastically permanent.

The proof of Theorem 2 is very similar to the proof Theorem1 we omit the details here to avoid repetition.

3. Summary

This paper is concerned with stochastic persistence of Stochastic Non-autonomous Lotka-Volterra Mutualism systems with Impulse Jump and Markov Switching. Two sufficient conditions for stochastic persistence are obtained. If impulse is bounded and Assumption 2-4 hold, Note that both the impulse and color noise have no impact on stochastic persistence of the population system (3). Some more interesting result deserve further investigation, we will attempt to investigate persistence in mean, weak-persistence, non-persistence, extinction and global attractivity of the system (3) at the next stage.

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