Geometric Power Lindley Poisson Distribution: Properties and Applications

Mahmoud M. Mansour

Department of Statistics, Mathematics and Insurance,
Benha University, Egypt
Mahmoud.mansour@fcom.bu.edu.eg

Mohammad Ahsanullah

Department of Management Sciences, Rider University
Lawrenceville, NJ 08648-3009
ahsan@rider.edu

Zohdy M. Nofal

Department of Statistics, Mathematics and Insurance,
Benha University, Egypt
dr_znofal@hotmail.com

Omar H. Khalaf

Department of Statistics, Mathematics and Insurance,
Benha University, Egypt

Abstract

In this Paper, a new four-parameter distribution motivated mainly by dealing with series-parallel system is introduced. Moments, conditional moments and moment generating function of the new distribution including are presented. Estimation of its parameters is studied. characterization of the new model is introduced.

Keywords: Lindley distribution; Geometric distribution; Maximum likelihood estimation; truncated Poisson distribution.

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1. Introduction

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. In view of it, extensive exertion has been used in the improvement of huge classes of standard likelihood circulations alongside applicable measurable procedures. Sankaran (1970) introduced the discrete Poisson Lindley distribution by compounding Poission and Lindley distributions. Ghitany et al. (2008) investigated the properties of the zero-truncated Poisson-Lindley distribution. Bakouch et al. (2012) extended. Lindley distribution by exponentiation. Zakerzadeh and Dolati (2010) introduced and analyzed a

three-parameter generalization of the Lindley distribution, which was used by Mahmoudi and Zakerzadeh (2010) to derive an extended version of the compound Poisson distribution. Shanker et al. (2013) introduced a two-parameter Lindley distribution in which the one-parameter is a particular case, for modeling waiting and survival time data. Ghitany et al. (2013) introduced a two-parameter power Lindley distribution (PL) and discussed its properties. Nadarajah et al. (2011) proposed a generalized Lindley distribution (GL) and provided a comprehensive account of its mathematical properties.

Nadarajah et al. (2013) introduced a two-parameter distribution which represent a general model by taking the probability density function and the cumulative distribution function of failure times to be given by $f(x)$ and $F(x)$, respectively. Its cdf is given by

$$G(x) = \frac{exp(-\lambda + \lambda F(x)) - e^{-\lambda}}{1 - e^{-\lambda} - \delta + \delta exp(-\lambda + \lambda F(x))},$$

(1)

for $x > 0, \lambda > 0 \ and \ 0 < \delta < 1$. The corresponding probability density function is,

$$g(x) = \frac{\lambda(1 - \delta)(1 - e^{-\lambda})f(x)exp(-\lambda + \lambda F(x))}{\{1 - e^{-\lambda} - \delta + \pi exp(-\lambda + \lambda F(x))\}^2}$$

(2)

Ghitany et al. (2013) introduced a new extension of the Lindley distribution called power Lindley distribution by its probability density function pdf

$$f(x) = \frac{\theta^2 \beta x^{\beta-1}}{\theta + 1}\left(1 + x^\beta\right)e^{-\theta x^\beta}, x > 0, \theta, \beta > 0.$$

(3)

The corresponding cumulative distribution function (cdf) is:

$$F(x) = 1 - \frac{\theta + 1 + \theta x^\beta}{\theta + 1}e^{-\theta x^\beta}, x > 0, \theta, \beta > 0.$$

(4)

The rest of the article is organized as follows. In Section 2, introduces the proposed new model according to geometric Poisson G-family. In Section 3, The Expansion for the pdf and the cdf Functions is derived. Moments, conditional moments and moment generating function of the new distribution including are presented in Section 4. In Section 5, we introduce Characterizations of the new model. In Section 6, we introduce the method of likelihood estimation as point estimation Finally, we fit the distribution to real data set to examine it.

2. Geometric Power Lindley Poisson Distribution

Using *f(x)* and *F(x)* as given in (3) and (4) in (1), we obtain

$$G(x) = \frac{exp\left(-\lambda\left(\frac{\theta + 1 + \theta x^\beta}{1 + \theta}\right)e^{-\theta x^\beta}\right) - e^{-\lambda}}{1 - e^{-\lambda} - \delta\left[1 - exp\left(-\lambda\left(\frac{\theta + 1 + \theta x^\beta}{1 + \theta}\right)e^{-\theta x^\beta}\right)\right]},$$

(5)

for $x > 0, \theta > 0, \lambda, \beta > 0 \ and \ 0 < \delta < 1$. The corresponding probability density function is,

$$g(x) = \frac{\lambda\theta^2(1-\delta)\beta x^{\beta-1}(1-e^{-\lambda})(1+x^\beta)}{(1+\theta)\left\{1-e^{-\lambda}-\delta\left[1-exp\left(-\lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right)\right]\right\}^2}$$

$$\times exp\left(-\theta x^\beta - \lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right) \tag{6}$$

for $x > 0, \theta > 0, \lambda, \beta > 0$ and $0 < \delta < 1$. We shall refer to the distribution given by (5) and (6) as the geometric power Lindley Poisson (*GPLP*). The parameters, λ, β and δ, control the shape. The parameter, θ, controls the scale. The failure rate function associated with (6) is given by

$$h(x) = \frac{\lambda\theta^2(1-\delta)\beta x^{\beta-1}(1-e^{-\lambda})(1+x^\beta)}{(1+\theta)\left\{1-e^{-\lambda}-\delta\left[1-exp\left(-\lambda+\lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right)\right]\right\}}$$

$$\times \frac{exp\left(-\theta x^\beta - \lambda + \lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right)}{\left(1-exp\left(-\lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right)\right)}. \tag{7}$$

Also, the reversed failure rate function

$$h_r(x) = \frac{\lambda\theta^2(1-\delta)(1-e^{-\lambda})(1+x)exp\left(-\theta x - \lambda\left(\frac{\theta+1+\theta x}{1+\theta}\right)e^{-\theta x}\right)}{(1+\theta)\left[exp\left(-\lambda\left(\frac{\theta+1+\theta x}{1+\theta}\right)e^{-\theta x}\right)-e^{-\lambda}\right]\{1-e^{-\lambda}-\delta[1-exp(-\lambda e^{-(\beta x)^\alpha})]\}}.$$

Figure 1 (a) and (b) provide some plots of the *GPLP* density curves for different values of the parameters λ, θ, β and δ.
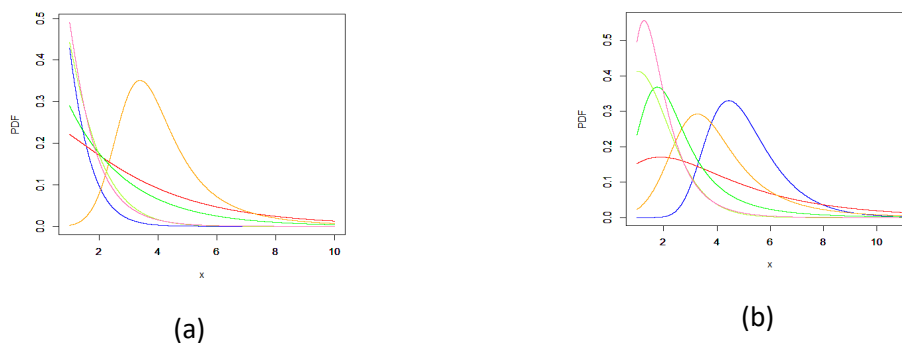

(a)   (b)

Fig. 1. Plots of the *GPLP* density function for some parameter values.(a) For λ=0.9, θ=2, β=1 and δ =0.1 (blue line), λ=2, θ=1 , β=0.7 and δ =0.2 (red line), λ=18, θ=1 , β=0.8 and δ =0.9 (orange line) λ=0.5, θ=1 , β=0.8 and δ =0.1 (green line), λ=0.5, θ=1 , β=1.2 and δ =0.5 (green yellow line), λ=4, θ=2 , β=0.8 and δ =0.5 (hot pink line) (b) For λ=30, θ=1, β=1 and δ =0.2 (blue line), λ=4, θ=1 , β=0.7 and δ =0.3 (red line), λ=10, θ=1 , β=0.7 and δ =0.3 (orange line) λ=8, θ=1 , β=0.8 and δ =0.9 (green line), λ=1.4, θ=1 , β=1.2 and δ =0.2 (green yellow line), λ=10, θ=2 , β=0.8 and δ =0.5 (hot pink line).

3. Expansion for the pdf and the cdf functions

In this section we give another expression for the pdf and the cdf functions using the Maclaurin and Binomial expansions for simplifying the pdf and the cdf forms.

3.1. *Expansion for the pdf function*

We can rewrite (6) as,

$$g(x) = \frac{\lambda\theta^2(1-\delta)\beta x^{\beta-1}(1-e^{-\lambda})(1+x^\beta)exp\left(-\theta x^\beta - \lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right)}{(1+\theta)(1-e^{-\lambda}-\delta)^2\left[1+\frac{\pi exp\left(-\lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right)}{1-e^{-\lambda}-\delta}\right]^2}. \tag{8}$$

We have

$$(1-z)^{-b} = \sum_{i=0}^{\infty}\binom{-b}{i}(-z)^i, \quad |z| < 1, \tag{9}$$

and

$$e^{-x} = \sum_{i=0}^{\infty}\frac{(-x)^i}{i!}, \tag{10}$$

Using (9), we can write (8) as

$$g(x) = \frac{\lambda\theta^2(1-\delta)\beta x^{\beta-1}(1-e^{-\lambda})(1+x^\beta)exp\left(-\theta x^\beta\right)}{(1+\theta)(1-e^{-\lambda}-\delta)^2} \times$$

$$\sum_{k=0}^{\infty}\binom{-2}{k}\left[\frac{\delta}{1-e^{-\lambda}-\delta}\right]^k exp\left(-\lambda(k+1)\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right). \tag{11}$$

Applying (10) to (11) for the term $exp\left(-\lambda(k+1)\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right)$, (11) can be written as:

$$g(x) = \frac{\lambda\theta^2(1-\delta)\beta x^{\beta-1}(1-e^{-\lambda})(1+x^\beta)exp\left(-\theta x^\beta\right)}{(1+\theta)(1-e^{-\lambda}-\delta)^2} \times$$

$$\sum_{k,j=0}^{\infty}\binom{-2}{k}\frac{(-1)^j}{j!}\left[\frac{\delta}{1-e^{-\lambda}-\delta}\right]^k \lambda^j(k+1)^j\left(1+\frac{\theta x^\beta}{1+\theta}\right)^j e^{-\theta j x^\beta}. \tag{12}$$

Applying (9) to 12) for the term $\left(1+\frac{\theta x^\beta}{1+\theta}\right)^j$, (12) can be written as:

$$g(x) = \frac{\lambda\theta^2(1-\delta)(1-e^{-\lambda})\beta}{(1+\theta)(1-e^{-\lambda}-\delta)^2} \sum_{k,j=0}^{\infty} \sum_{i=0}^{j} \binom{j}{i}\binom{-2}{k}\frac{(-1)^j\theta^i}{j!\,(1+\theta)^i} \times$$

$$\left[\frac{\pi}{1-e^{-\lambda}-\pi}\right]^k \lambda^j(k+1)^j(1+x^\beta)x^{i\beta}x^{\beta-1}e^{-(j+1)\theta x^\beta}. \tag{13}$$

The pdf of *GPLP* distribution can then be represented as:

$$g(x) = \sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i}\,(1+x^\beta)x^{\beta(i+1)-1}e^{-(j+1)\theta x^\beta}, \tag{14}$$

Where $A_{k:i}$ is a constant term given by:

$$A_{k:i} = \binom{-2}{k}\binom{j}{i}\frac{\theta^{2+i}(1-\delta)(1-e^{-\lambda})(-1)^j\delta^k\,\lambda^{j+1}(k+1)^j\beta}{(1+\theta)^{i+1}(1-e^{-\lambda}-\delta)^{k+2}\,j!}.$$

3.2. *Expansion for the cdf function*

We can rewrite (5) as,

$$G(x) = \frac{exp\left(-\lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right) - e^{-\lambda}}{(1-e^{-\lambda}-\delta)\left[1+\dfrac{\pi\,exp\left(-\lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right)}{1-e^{-\lambda}-\delta}\right]}$$

And applying the expansion in (9), the cdf function of the GPLP distribution can be written as:

$$G(x) = \sum_{k=0}^{\infty}\binom{-1}{k}\frac{\pi^k\left[exp\left(-(k+1)\lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right) - exp\left(-\lambda - k\lambda\left(\frac{\theta+1+\theta x^\beta}{1+\theta}\right)e^{-\theta x^\beta}\right)\right]}{(1-e^{-\lambda}-\delta)^{k+1}}.$$

4. Statistical Properties

In this section, moments, conditional moments, Moment Generating Function of the *GPLP* distribution.

4.1. *Moments*

The rth non-central moments or (moments about the origin) of the *GPLP* under using equation (6) is given by:

$$\mu_r = E(X^r) = \int\limits_0^\infty X^r g(x)\mathrm{d}x,$$

$$\mu_r = \int\limits_0^\infty x^r \left[\sum_{k,j=0}^\infty \sum_{i=0}^j A_{k:i} \, (1 + x^\beta) x^{\beta(i+1)-1} e^{-(j+1)\theta x^\beta} \right] \mathrm{d}x,$$

Then

$$\mu_r = \sum_{k,j=0}^\infty \sum_{i=0}^j A_{k:i} \left[\frac{\Gamma(i + \frac{r}{\beta} + 1)}{\beta \, [(j+1)\theta]^{(i+\frac{r}{\beta}+1)}} \right] \left[1 + \frac{i + \frac{r}{\beta} + 1}{(j+1)\theta} \right],$$

$$E(X|X \le x) = \frac{1}{\beta \, G(x)} \sum_{k,j=0}^\infty \sum_{i=0}^j A_{k:i} \left[\frac{\Gamma^c_{(j+1)\theta x^\beta}(i + \frac{r}{\beta} + 1)}{[(j+1)\theta]^{(i+\frac{r}{\beta}+1)}} + \frac{\Gamma^c_{(j+1)\theta x^\beta}(i + \frac{r}{\beta} + 2)}{[(j+1)\theta]^{(i+\frac{r}{\beta}+2)}} \right],$$

$$E(X|X \ge x) = \frac{1}{\beta[1-G(x)]} \sum_{k,j=0}^\infty \sum_{i=0}^j A_{k:i} \left[\frac{\Gamma^c_{(j+1)\theta x^\beta}(i + \frac{r}{\beta} + 1)}{[(j+1)\theta]^{(i+\frac{r}{\beta}+1)}} + \frac{\Gamma^c_{(j+1)\theta x^\beta}(i + \frac{r}{\beta} + 2)}{[(j+1)\theta]^{(i+\frac{r}{\beta}+2)}} \right],$$

where $\Gamma_\alpha(n) = \int_0^\alpha x^{n-1} e^{-x} dx$ and $\Gamma^c_\alpha(n) = \int_\alpha^\infty x^{n-1} e^{-x} dx$.

4.2. *The Moment Generating Function*

The moment generating function, $M_x(t)$, can be easily obtained as:

$$M_x(t) = \int\limits_0^\infty e^{tx} g(x)\mathrm{d}x,$$

$$M_x(t) = \int\limits_0^\infty e^{tx} \sum_{k,j=0}^\infty \sum_{i=0}^j A_{k:i} \, (1 + x^\beta) x^{\beta(i+1)+r-1} e^{-(j+1)\theta x^\beta} \mathrm{d}x,$$

then, the moment generating function of the *GPLP* distribution is given by,

$$M_x(t) = \sum_{k,j=0}^\infty \sum_{i=0}^j \frac{A_{k:i}}{[\theta(j+1) - t]^{i+2}} \{(\theta(j+1) - t)\Gamma(i+1) + \Gamma(i+2)\},$$

then, the moment generating function of the *GPLP* distribution is given by,

$$M_x(t) = \sum_{k,j=0}^{\infty} \sum_{i=0}^{j} \frac{A_{k:i}}{[(j+1)\theta]^{\left(i+\frac{r}{\beta}+2\right)}} \left\{ ((j+1)\theta - t)\Gamma\left(i+\frac{1}{\beta}+1\right) + \Gamma\left(i+\frac{1}{\beta}+2\right) \right\}.$$

5. Characterization

To prove the main theorem, we need the following two Lemmas

Lemma 1.

Suppose that X is an absolutely continuous random variable with cdf $F(x)$ with $F(0) = 0$ and $F(x) > 0$ for all $x > 0$. We assume that the pdf of X as $f(x)$ and $f'(x)$ exists for *all* $x > 0$., For a continuous function $g(x)$ on $0 < x < \infty$ with finite $E(g(x))$ .if $E(g(x)|x \le x) = h_1(x)r(x)$, where $h_1(x)$ is a differential function in $x > 0$ and $r(x) = \frac{f(x)}{F(x)}$, then $f(x) = c \, exp\left(-\int_0^x \frac{g(u)-h(u)}{h(u)} du\right)$ and c is determned by the condition $\int_0^{\infty} f(x)dx = 1$.

Proof.

$$h_1(x) = \frac{\int_0^x g(u)f(u)du}{f(x)}$$

and

$$\int_0^x g(u)f(u)du = f(x)\,h_1(x)$$

Differentiating the above expression, we obtain

$$-g(x)f(x) = f(x)h_1'(x) + \acute{f}(x)h_1(x)$$

On simplification, we have

$$\frac{\acute{f}(x)}{f(x)} = \frac{g(x) + h_1'(x)}{h_1(x)}$$

Integrating both sides of the above equation, we obtain

$$f(x) = c \, exp\left(-\int_0^x \frac{g(x) + h_1'(x)}{h_1(x)}dx\right)$$

and c is determned by the condition $\int_0^{\infty} f(x)dx = 1$.

Lemma 2.

Suppose that X is an absolutely continuous random variable with cdf $F(x)$ with $F(0) = 0$ and $F(x) > 0$ for all $x > 0$. We assume that the pdf of X as $f(x)$ and $f'(x)$ exists for all $x > 0$., For a continuous function $g(x)$ on $0 < x < \infty$ with finite $E(g(x))$ .if $E(g(x)|x \geq x) = h(x)r(x)$, where $h(x)$ is a differential function in $x > 0$ and , $r(x) = \frac{f(x)}{1-F(x)}$, then $f(x) = c\ exp\left(-\int \frac{g(x)-\acute{h}(x)}{h(x)}dx\right)$ and c is determned by the condition $\int_0^x f(x)dx = 1$.

Proof.

$$h(x) = \frac{\int_x^\infty g(u)f(u)du}{f(x)}$$

and

$$\int_x^\infty g(u)f(u)du = f(x)\,h(x)$$

Differentiating the above expression, we obtain

$$g(x)f(x) = f(x)h'(x) + f'(x)h(x)$$

On simplification, we have

$$\frac{\acute{f}(x)}{f(x)} = -\frac{g(x) - h'(x)}{h(x)}$$

Integrating both sides of the above equation, we obtain

$$f(x) = c\ exp\left(-\int_0^x \frac{g(u) - h'(u)}{h(u)}du\right)$$

and c is determned by the condition $\int_0^\infty f(x)dx = 1$.

Theorem 1.

Suppose that X is an absolutely continuous random variable with cdf $G(x)$ with $G(0) = 0$ and $G(x) > 0$ for all $x > 0$. We assume that the pdf of X as $g(x)$ and $g'(x)$ exists for all $x > 0$.

We assume $E(X^r)$ exists for $r \geq 1$. then $E(X^r|X \leq x) = h_0(x)\tau(x)$, where $\tau(x) = \frac{g(x)}{G(x)}$ and $h_0(x) = \frac{A(x)}{B(x)}$,

where $A(x) = \sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left[\frac{\Gamma_{(j+1)\theta x^\beta}(i+\frac{r}{\beta}+1)}{\beta\left[(j+1)\theta\right]^{(i+\frac{r}{\beta}+1)}} + \frac{\Gamma_{(j+1)\theta x^\beta}(i+\frac{r}{\beta}+2)}{\beta\left[(j+1)\theta\right]^{(i+\frac{r}{\beta}+2)}} \right]$

$B(x) = g(x)$ and $A_{k:i}$ is given before

if and only if $g(x) = \sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} (1 + x^\beta) x^{\beta(i+1)-1} e^{-(j+1)\theta x^\beta}$

Proof.

We have

$$g(x) h_0(x) = \int_0^x u^r g(u) du$$

$$= \sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left[\frac{\Gamma_{(j+1)\theta x^\beta}(i + \frac{r}{\beta} + 1)}{\beta[(j+1)\theta]^{(i+\frac{r}{\beta}+1)}} + \frac{\Gamma_{(j+1)\theta x^\beta}(i + \frac{r}{\beta} + 2)}{\beta[(j+1)\theta]^{(i+\frac{r}{\beta}+2)}} \right]$$

$$= A(x)$$

where $\Gamma_\alpha(\beta) = \int_\alpha^\infty x^{\beta-1} e^{-x} dx$

Suppose that $h_0(x) = \frac{A(x)}{g(x)}$, then

$$h_0'(x) = x^r - \frac{A(x)}{\left(g(x)\right)^2} g'(x)$$

$$h_0'(x) = x^r - h_0(x) \frac{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} x^{\beta(i+1)-2} + \left(x^{\beta(i+2)-1}\right) x^{\beta(i+2)-2} - (j+1)\theta\beta x^{\beta-1}\right) e^{-(j+1)\theta x^\beta}}{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} + x^{\beta(i+2)-1}\right) e^{-(j+1)\theta x^\beta}}$$

Thus

$$\frac{x^r - h_0'(x)}{h_0(x)} = \frac{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} x^{\beta(i+1)-2} + \left(x^{\beta(i+2)-1}\right) x^{\beta(i+2)-2} - (j+1)\theta\beta x^{\beta-1}\right) e^{-(j+1)\theta x^\beta}}{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} + x^{\beta(i+2)-1}\right) e^{-(j+1)\theta x^\beta}}$$

Integrating both sides of the above equation we obtain

$$g(x) = c \, exp \int \left(\frac{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} x^{\beta(i+1)-2} + \left(x^{\beta(i+2)-1}\right) x^{\beta(i+2)-2} - (j+1)\theta\beta x^{\beta-1}\right) e^{-(j+1)\theta x^\beta}}{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} + x^{\beta(i+2)-1}\right) e^{-(j+1)\theta x^\beta}} \right) dx$$

$$= c \sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} + x^{\beta(i+2)-1}\right) e^{-(j+1)\theta x^\beta}, \text{ where c is a constant}$$

and

$$\frac{1}{c} = \int_0^\infty \sum_{k,j=0}^\infty \sum_{i=0}^j A_{k:i} \left(x^{\beta(i+1)-1} + x^{\beta(i+2)-1} \right) e^{-(j+1)\theta x^\beta}.$$

Theorem 2.

Suppose that X is an absolutely continuous random variable with cdf $G(x)$ with $G(0) = 0$ and $G(x) > 0$ for all $x > 0$. We assume that the pdf of X as $g(x)$ and $g'(x)$ exists for all $x > 0$.

We assume $E(X^{rm})$ exists for $r \geq 1$.then $E(X^r|X \geq x) = h_1(x)\tau(x)$, where $\tau(x) = \frac{g(x)}{1-G(x)}$ and $h_1(x) = \frac{A_1(x)}{B(x)}$,

where $A_1(x) = \frac{1}{\beta[1-G(x)]} \sum_{j=0}^\infty \sum_{i=0}^j A_{k:i} \left[\frac{\Gamma^c_{(j+1)\theta x^\beta}(i+\frac{r}{\beta}+1)}{[(j+1)\theta]^{(i+\frac{r}{\beta}+1)}} + \frac{\Gamma^c_{(j+1)\theta x^\beta}(i+\frac{r}{\beta}+2)}{[(j+1)\theta]^{(i+\frac{r}{\beta}+2)}} \right]$

$B(x) = g(x)$ and $A_{k:i}$ is given before

if and only if $g(x) = \sum_{k,j=0}^\infty \sum_{i=0}^j A_{k:i} (1 + x^\beta) x^{\beta(i+1)-1} e^{-(j+1)\theta x^\beta}$

Proof.

We have

$$g(x)h_1(x) = \int_x^\infty u^r g(u)du$$

$$= \frac{1}{\beta[1 - G(x)]} \sum_{k,j=0}^\infty \sum_{i=0}^j A_{k:i} \left[\frac{\Gamma^c_{(j+1)\theta x^\beta}(i + \frac{r}{\beta} + 1)}{[(j + 1)\theta]^{(i+\frac{r}{\beta}+1)}} + \frac{\Gamma^c_{(j+1)\theta x^\beta}(i + \frac{r}{\beta} + 2)}{[(j + 1)\theta]^{(i+\frac{r}{\beta}+2)}} \right]$$

$$= A_1(x)$$

where $\Gamma_\alpha(\beta) = \int_\alpha^\infty x^{\beta-1} e^{-x} dx$

Thus $h_1(x) = \frac{A_1(x)}{g(x)}$

Suppose that $h_1(x) = \frac{A_1(x)}{g(x)}$, then

$$h_1'(x) = -x^r + \frac{A_1(x)\, g'(x)}{\left(g(x)\right)^2}$$

$$= -x^r + h_1(x) \frac{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} x^{\beta(i+1)-2} + \left(x^{\beta(i+2)-1} \right) x^{\beta(i+2)-2} - (j+1)\theta\beta x^{\beta-1} \right) e^{-(j+1)\theta x^{\beta}}}{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} + x^{\beta(i+2)-1} \right) e^{-(j+1)\theta x^{\beta}}}$$

Thus

$$\frac{x^r + \acute{h}_1(x)}{h_1(x)} = \frac{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} x^{\beta(i+1)-2} + \left(x^{\beta(i+2)-1} \right) x^{\beta(i+2)-2} - (j+1)\theta\beta x^{\beta-1} \right) e^{-(j+1)\theta x^{\beta}}}{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} + x^{\beta(i+2)-1} \right) e^{-(j+1)\theta x^{\beta}}}$$

Integrating both sides of the above equation we obtain

$$g(x) = c\, exp \int \left(\frac{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} x^{\beta(i+1)-2} + \left(x^{\beta(i+2)-1} \right) x^{\beta(i+2)-2} - (j+1)\theta\beta x^{\beta-1} \right) e^{-(j+1)\theta x^{\beta}}}{\sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} + x^{\beta(i+2)-1} \right) e^{-(j+1)\theta x^{\beta}}} \right) dx$$

$$= c \sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} + x^{\beta(i+2)-1} \right) e^{-(j+1)\theta x^{\beta}}, \text{ where c is a constant}$$

and

$$\frac{1}{c} = \int_{0}^{\infty} \sum_{k,j=0}^{\infty} \sum_{i=0}^{j} A_{k:i} \left(x^{\beta(i+1)-1} + x^{\beta(i+2)-1} \right) e^{-(j+1)\theta x^{\beta}}.$$

6. Estimation of the Parameters

In this section we introduce the method of likelihood to estimate the parameters. The maximum likelihood estimators (MLEs) for the parameters of the geometric power Lindley Poisson distribution GPLP$(\lambda, \theta, \delta, \beta)$ is discussed in this section. Consider the random sample $x_1, x_2, \ldots, x_n$ of size n from GPLP $(\lambda, \theta, \delta, \beta)$ with probability density function in (6), then the likelihood function can be expressed as follows

$$\ell = \frac{\lambda^n \theta^{2n} (1-\delta)^n (1-e^{-\lambda})^n \beta^n \prod_{i=1}^{n} x_i^{\beta-1} \prod_{i=1}^{n} (1+x_i^{\beta})\, exp\left(-\theta \sum_{i=1}^{n} x_i^{\beta} - \lambda \sum_{i=1}^{n} \left(\frac{\theta+1+\theta x_i^{\beta}}{1+\theta} \right) e^{-\theta x_i} \right)}{(1+\theta)^n \prod_{i=1}^{n} \left\{ 1 - e^{-\lambda} - \delta \left[1 - exp\left(-\lambda \left(\frac{\theta+1+\theta x_i^{\beta}}{1+\theta} \right) e^{-\theta x_i^{\beta}} \right) \right] \right\}^2}. \tag{15}$$

Hence, the log-likelihood function, $\mathcal{L}$, becomes:

$$\mathcal{L} = n \ln \lambda + 2n \ln \theta + n \ln(1-\delta) + n \ln(1 - e^{-\lambda}) + n \ln\beta + \sum_{i=1}^{n}(\beta - 1) \ln x_i$$

$$-n \ln(1+\theta) + \sum_{i=1}^{n}(1 + x_i^{\beta}) - \theta \sum_{i=1}^{n} x_i^{\beta} - \lambda \sum_{i=1}^{n}\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}$$

$$-2\sum_{i=1}^{n} \ln\left[1 - e^{-\lambda} - \delta\left[1 - exp\left(-\lambda\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}\right)\right]\right]$$

(16)

Therefore, the MLEs of λ, θ, δ *and* β must satisfy the following equations:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{n}{\lambda} + \frac{n}{e^{\lambda} - 1} - \sum_{i=1}^{n}\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}$$

$$-2\sum_{i=1}^{n}\frac{e^{-\lambda} - \delta\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) exp\left(-\theta x_i^{\beta} - \lambda\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}\right)}{\left[1 - e^{-\lambda} - \delta\left[1 - exp\left(-\lambda\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}\right)\right]\right]},$$

(17)

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n} x_i^{\beta} + \frac{\lambda\theta}{(1+\theta)^2}\sum_{i=1}^{n}\left[(2+\theta)x_i^{\beta} + (1+\theta)x_i^{\beta^2}\right] e^{-\theta x_i^{\beta}}$$

$$-\sum_{i=1}^{n}\frac{2\lambda\theta\delta \, exp\left(-\theta x_i^{\beta} - \lambda\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}\right)\left[(2+\theta)x_i^{\beta} + (1+\theta)x_i^{\beta^2}\right]}{(1+\theta)^2\left[1 - e^{-\lambda} - \delta\left[1 - exp\left(-\lambda\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}\right)\right]\right]},$$

(18)

$$\frac{\partial \mathcal{L}}{\partial \pi} = \frac{-n}{(1-\delta)} + 2\sum_{i=1}^{n}\frac{1 - exp\left(-\lambda\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}\right)}{1 - e^{-\lambda} - \delta\left[1 - exp\left(-\lambda\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}\right)\right]},$$

19)

and

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln x_i + (1-\theta)\sum_{i=1}^{n} x_i^{\beta} \ln x_i - \left(\frac{\theta x_i^{\beta} - \theta}{1+\theta}\right) exp(-\theta x_i^{\beta} \, \theta x_i^{\beta} \, \ln x_i)$$

$$-2\sum_{i=1}^{n}\frac{\pi \, exp\left(-\lambda\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}\right)\left(\frac{\theta x_i^{\beta} - \theta}{1+\theta}\right) exp(-\theta x_i^{\beta} \, \theta x_i^{\beta} \, \ln x_i)}{\left[1 - e^{-\lambda} - \delta\left[1 - exp\left(-\lambda\left(\frac{\theta + 1 + \theta x_i^{\beta}}{1+\theta}\right) e^{-\theta x_i^{\beta}}\right)\right]\right]}$$

(20)

The maximum likelihood estimator $\underline{\hat{\vartheta}} = (\hat{\lambda}, \hat{\theta}, \hat{\delta}, \hat{\beta})$ of $\underline{\vartheta} = (\lambda, \theta, \delta, \beta)$ is obtained by solving the nonlinear system of equations (17) through (20). It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function.

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