Numerical Method of Elastodynamic Equation with Curvilinear grid

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Abstract. For the cross sectional profile of axisymmetric body with curve, the numerical solution of elastodynamic equation is difficult in cylindrical coordinates. In order to realize the dynamic simulation of axisymmetric body with curve cross section, based on the Lax-Wendroff differential theory, a numerical solution method of elastodynamic equation was presented with curvilinear grid. The dynamic response of impact system of underwater pile driving hammer was simulated, which results were reasonably explained by the stress wave theory. The numerical solution method presented is valid.

1 Introduction

The research on numerical solution of axisymmetric elastodynamic equations started earlier, the basic finite difference method is developed as early as the Second World War [1]. Berthold and Karnes [2] studied two aluminium rods axial impact by two dimensional finite difference method, and they obtained reasonable numerical results base on the artificial viscosity method by Von Neumann and Richtmeyer [3]. In 1995, Lin and Ballmann [4] developed a Lax - Wendroff difference scheme and used it to obtain numerical solution of axisymmetric stress wave in regular elastomer. Valsamos [5] presented an explicit finite element method to simulate longitudinal wave propagation in rod. The numerical results were consistent with analytic solution of Pochhammer-Chree equations. The above mentioned studies are for axisymmetric body with regular cross-section. However, there have been few studies irregular cross-section. This work presents a numerical scheme for axisymmetric body with smooth curve cross-section. The numerical scheme is based on Lax-Wendroff finite difference method. Impact system of pile driving hammer is successfully simulated by the numerical scheme.

2 Axisymmetric elastic dynamic equations

During the impact between two axisymmetric elastomers, if the deformation of the two elastomers is axisymmetric deformation, and the stress wave is untwisted stress wave in the two elastomers, then the dynamic response of the two elastomers is typical spatial axisymmetric elastic dynamics problem.

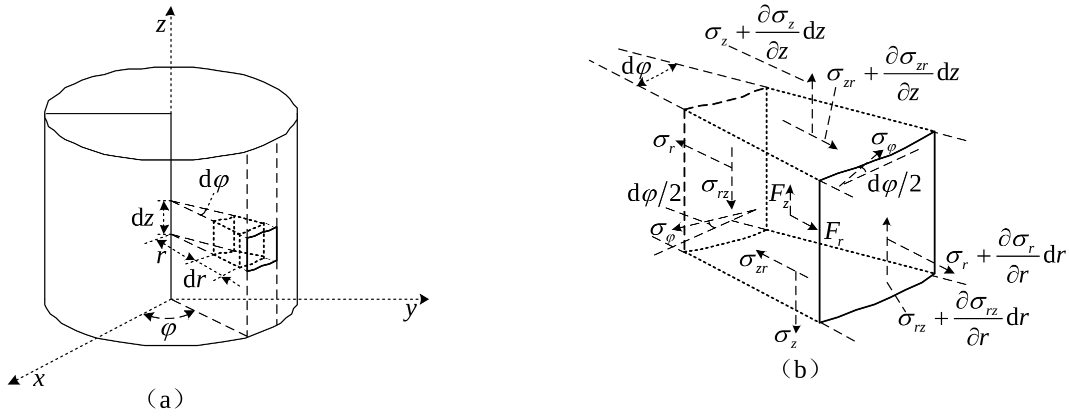

Figure 1. Analysing forces acted on the infinitesimal body for axisymmetric problem.

Cylindrical coordinate system is established, as shown in Figure 1(a). Take an infinitesimal body from the axisymmetric body. The stress on the infinitesimal body is analysed, as shown in Figure 1(b). σ_z is average normal stress on the lower plane, σ_r is average radial normal stress on the inner cylinder face, σ_φ is circumferential average normal stress, σ_{rz} is shear stress along the z axis on the inner cylinder face, F_r and F_z are body force, w is axial displacement, u is radial displacement. For axisymmetric problem, the value of σ_{rz} is equal to σ_{zr}, $\sigma_{\varphi r}$ and $\sigma_{r\varphi}$ are not exist. Equation (1) is the elasticity formulas for axisymmetric problem.

$$\begin{cases} \sigma_z = \dfrac{E}{1+\upsilon}\left(\dfrac{1-\upsilon}{1-2\upsilon}\dfrac{\partial w}{\partial z} + \dfrac{\upsilon}{1-2\upsilon}\dfrac{\partial u}{\partial r} + \dfrac{\upsilon}{1-2\upsilon}\dfrac{u}{r}\right) \\ \sigma_r = \dfrac{E}{1+\upsilon}\left(\dfrac{\upsilon}{1-2\upsilon}\dfrac{\partial w}{\partial z} + \dfrac{1-\upsilon}{1-2\upsilon}\dfrac{\partial u}{\partial r} + \dfrac{\upsilon}{1-2\upsilon}\dfrac{u}{r}\right) \\ \sigma_\varphi = \dfrac{E}{1+\upsilon}\left(\dfrac{\upsilon}{1-2\upsilon}\dfrac{\partial w}{\partial z} + \dfrac{\upsilon}{1-2\upsilon}\dfrac{\partial u}{\partial r} + \dfrac{1-\upsilon}{1-2\upsilon}\dfrac{u}{r}\right) \\ \sigma_{rz} = \dfrac{E}{2(1+\upsilon)}\left(\dfrac{\partial w}{\partial r} + \dfrac{\partial u}{\partial z}\right) \\ \dfrac{\partial \sigma_z}{\partial z} + \dfrac{\partial \sigma_{rz}}{\partial r} + \dfrac{\sigma_{rz}}{r} + F_z = 0 \\ \dfrac{\partial \sigma_r}{\partial r} + \dfrac{\partial \sigma_{rz}}{\partial z} + \dfrac{\sigma_r - \sigma_\varphi}{r} + F_r = 0 \end{cases} \quad (1)$$

The components of stresses are not only the functions of z and r but also time t for axisymmetric elastodynamic problem, equation (1) is still applying [6]. By the D'Alembert principle, the inertia forces on the unit volume are

$$-\rho\frac{\partial v_r}{\partial t},\ -\rho\frac{\partial v_z}{\partial t}.$$

Where, v_z is axial velocity; v_r is radial velocity; ρ is the mass density; F_r and F_z are going to be zero when gravity is not considered. Take the derivative of σ_z, σ_r, σ_φ and σ_{rz} with respect to t. Equation (2) are the elastodynamic formulas for axisymmetric problem.

$$\begin{cases} \dfrac{\partial \sigma_z}{\partial t} = \rho c_l^2 \dfrac{\partial v_z}{\partial z} + \eta\dfrac{\partial v_r}{\partial r} + \eta\dfrac{v_r}{r} \\ \dfrac{\partial \sigma_r}{\partial t} = \eta\dfrac{\partial v_z}{\partial z} + \rho c_l^2 \dfrac{\partial v_r}{\partial r} + \eta\dfrac{v_r}{r} \\ \dfrac{\partial \sigma_\varphi}{\partial t} = \eta\dfrac{\partial v_z}{\partial z} + \eta\dfrac{\partial v_r}{\partial r} + \rho c_l^2\dfrac{v_r}{r} \\ \dfrac{\partial \sigma_{rz}}{\partial t} = \rho c_s^2 \dfrac{\partial v_r}{\partial z} + \rho c_s^2 \dfrac{\partial v_z}{\partial r} \\ \dfrac{\partial v_z}{\partial t} = \dfrac{1}{\rho}\dfrac{\partial \sigma_z}{\partial z} + \dfrac{1}{\rho}\dfrac{\partial \sigma_{rz}}{\partial r} + \dfrac{1}{\rho}\dfrac{\sigma_{rz}}{r} \\ \dfrac{\partial v_r}{\partial t} = \dfrac{1}{\rho}\dfrac{\partial \sigma_{rz}}{\partial z} + \dfrac{1}{\rho}\dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{\rho}\dfrac{\sigma_r - \sigma_\varphi}{r} \end{cases} \quad (2)$$

Where, $c_l = \sqrt{E/\rho}$ is longitudinal wave velocity of material; $c_s = \sqrt{E/2\rho(1+\upsilon)}$ is transversal waves velocity of material; $\eta=(\rho c_l^2 - \rho c_s^2)$ is a coefficient. Equation (3) is matrix form of Equation (2)

$$\frac{\partial \boldsymbol{U}}{\partial t} = \boldsymbol{A}_1\frac{\partial \boldsymbol{U}}{\partial z} + \boldsymbol{A}_2\frac{\partial \boldsymbol{U}}{\partial r} + \boldsymbol{A}_3\boldsymbol{U} \quad (3)$$

Where, $\boldsymbol{U} = (\sigma_z, \sigma_r, \sigma_\varphi, \sigma_{rz}, v_z, v_r)^T$; The coefficient matrix $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$ is:

$$\boldsymbol{A}_1=\begin{pmatrix}0&0&0&0&\rho c_l^2&0\\0&0&0&0&\eta&0\\0&0&0&0&\eta&0\\0&0&0&0&0&\rho c_s^2\\1/\rho&0&0&0&0&0\\0&0&0&1/\rho&0&0\end{pmatrix},\ \boldsymbol{A}_2=\begin{pmatrix}0&0&0&0&0&\eta\\0&0&0&0&0&\rho c_l^2\\0&0&0&0&0&\eta\\0&0&0&0&\rho c_s^2&0\\0&0&0&1/\rho&0&0\\0&1/\rho&0&0&0&0\end{pmatrix},$$

$$\boldsymbol{A}_3=\begin{pmatrix}0&0&0&0&0&\eta/r\\0&0&0&0&0&\eta/r\\0&0&0&0&0&\rho c_l^2/r\\0&0&0&0&0&0\\0&0&0&1/\rho r&0&0\\0&1/\rho r&-1/\rho r&0&0&0\end{pmatrix}.$$

According to the initial conditions and boundary conditions, solve the equation (2) can obtain its dynamic response. Equation (2) is the system of hyperbolic partial differential equations with a source term. The analytical solution of equation (2) can be obtained only under simple geometric boundary and material constitutive relation and loading condition[7].

3 Numerical method of elastodynamic equation with curvilinear grids

Equation (2) cannot be solved when the cross-section outline of the axially symmetric objects contain curve, then the equation (2) only be solved numerically. Numerical method need divide mesh for the cross-section outline of the axially symmetric objects. To divide mesh under rectangular coordinate system will not obtain satisfying grid, and the description of boundary conditions will inaccuracy. Curvilinear grids under curvilinear coordinates will clearly describe the cross-section outline of the axially symmetric objects. In order to obtain dynamic response of the irregular axially symmetric objects need divide the cross-section under the curvilinear coordinates. However, it needs to derive elastodynamic equation under the curvilinear coordinates. The form of the derived equation is complex. The numerical computational scheme of the derived equation is hard to get. A numerical method of to obtain the dynamic response of the irregular axially symmetric objects is presented below.

First, mesh under the body-fitted coordinate system. Then the node coordinates are obtained. Second, transform the curve coordinates of the node to the rectangular coordinates. Third, use the difference scheme under rectangular coordinate system to calculate.

Coordinate transformation method is presented, as shown in Figure 2. Dot and diamond express the node and element, respectively. *x*, *y* are the axes of the body-fitted coordinate system. *n* is sequence numbers of *x* coordinate. *m* is sequence numbers of *y* coordinate. *‘r* and *‘z* are the *r* coordinate and *z* coordinate of the node under the rectangular coordinate system.

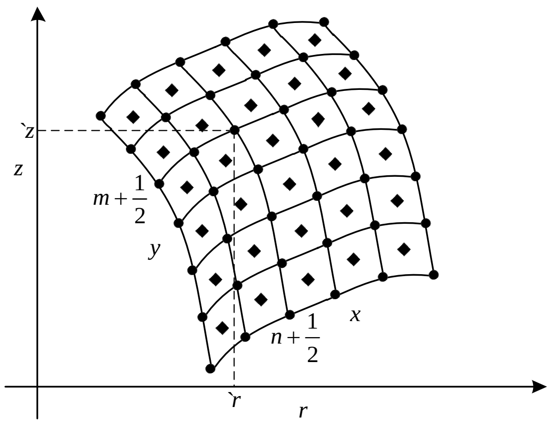

Figure 2. Principle of coordinate transformation.

As shown in Figure 2, the curve grids are uneven, the spatial step length of elements will be different, and the edge of element will not parallel or vertical with coordinate axis. In order to use the difference scheme under the rules grid, need to acquire the equivalent step length of each element and node. The influence of element size and step length are considered in the difference scheme.Element area $S^e_{m,n}$ and node area is introduced to the difference scheme. The node area is $S_{(m+1/2,n+1/2)}=(S^e_{m,n}+S^e_{m+1,n}+S^e_{m,n+1}+S^e_{m+1,n+1})/4$. The first step of the difference scheme is:

$$\begin{aligned} \boldsymbol{U}^{t+\frac{1}{2}}_{m+\frac{1}{2},n+\frac{1}{2}} = \frac{1}{4S_{m+\frac{1}{2},n+\frac{1}{2}}}(& S^e_{m,n}\boldsymbol{U}^t_{m,n} + S^e_{m+1,n}\boldsymbol{U}^t_{m+1,n} + \\ & S^e_{m,n+1}\boldsymbol{U}^t_{m,n+1} + S^e_{m+1,n+1}\boldsymbol{U}^t_{m+1,n+1}) + \\ & \frac{\Delta t}{4}\boldsymbol{A}_1(\frac{\boldsymbol{U}^t_{m+1,n+1} - \boldsymbol{U}^t_{m,n}}{{}^1\Delta r_{m+\frac{1}{2},n+\frac{1}{2}}} - \frac{\boldsymbol{U}^t_{m+1,n} - \boldsymbol{U}^t_{m,n+1}}{{}^2\Delta r_{m+\frac{1}{2},n+\frac{1}{2}}}) - \\ & \frac{\Delta t}{4}\boldsymbol{A}_2(\frac{\boldsymbol{U}^t_{m+1,n+1} - \boldsymbol{U}^t_{m,n}}{{}^1\Delta z_{m+\frac{1}{2},n+\frac{1}{2}}} - \frac{\boldsymbol{U}^t_{m+1,n} - \boldsymbol{U}^t_{m,n+1}}{{}^2\Delta r_{m+\frac{1}{2},n+\frac{1}{2}}}) + \\ & \frac{\Delta t}{8S_{m+\frac{1}{2},n+\frac{1}{2}}}\boldsymbol{A}_3(S^e_{m,n}\boldsymbol{U}^t_{m,n} + S^e_{m+1,n}\boldsymbol{U}^t_{m+1,n} + \\ & S^e_{m,n+1}\boldsymbol{U}^t_{m,n+1} + S^e_{m+1,n+1}\boldsymbol{U}^t_{m+1,n+1}) \end{aligned} \tag{4}$$

The second step of the difference scheme is Equation(5). Where, ${}^1\Delta r_{m,n}$ and ${}^2\Delta r_{m,n}$ are the equivalent radial step length of element; ${}^1\Delta z_{m,n}$ and ${}^2\Delta z_{m,n}$ are the the equivalent axial step length of element; ${}^1\Delta r_{m+1/2,n+1/2}$ and ${}^2\Delta r_{m+1/2,n+1/2}$ are the equivalent radial step length of node; ${}^1\Delta z_{m+1/2,n+1/2}$ and ${}^2\Delta z_{m+1/2,n+1/2}$ are the the equivalent axial step length of node. Because the size of each curve element is different, the equivalent step length of each element and node need to be calculated, respectively. Principle of equivalence is to keep the area of element and node unchanged.

$$\begin{aligned} \boldsymbol{U}^{t+1}_{m,n} = [& \boldsymbol{A}_1(\frac{\boldsymbol{U}^{t+\frac{1}{2}}_{m+\frac{1}{2},n+\frac{1}{2}} - \boldsymbol{U}^{t+\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}}}{{}^1\Delta r_{m,n}} - \frac{\boldsymbol{U}^{t+\frac{1}{2}}_{m+\frac{1}{2},n-\frac{1}{2}} - \boldsymbol{U}^{t+\frac{1}{2}}_{m-\frac{1}{2},n+\frac{1}{2}}}{{}^2\Delta r_{m,n}}) - \\ & \boldsymbol{A}_2(\frac{\boldsymbol{U}^{t+\frac{1}{2}}_{m+\frac{1}{2},n+\frac{1}{2}} - \boldsymbol{U}^{t+\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}}}{{}^1\Delta z_{m,n}} - \frac{\boldsymbol{U}^{t+\frac{1}{2}}_{m+\frac{1}{2},n-\frac{1}{2}} - \boldsymbol{U}^{t+\frac{1}{2}}_{m-\frac{1}{2},n+\frac{1}{2}}}{{}^2\Delta z_{m,n}}) + \\ & \frac{1}{2S_{m,n}}\boldsymbol{A}_3(S_{m+\frac{1}{2},n+\frac{1}{2}}\boldsymbol{U}^{t+\frac{1}{2}}_{m+\frac{1}{2},n+\frac{1}{2}} + S_{m+\frac{1}{2},n-\frac{1}{2}}\boldsymbol{U}^{t+\frac{1}{2}}_{m+\frac{1}{2},n-\frac{1}{2}} + \\ & S_{m-\frac{1}{2},n+\frac{1}{2}}\boldsymbol{U}^{t+\frac{1}{2}}_{m-\frac{1}{2},n+\frac{1}{2}} + S_{m-\frac{1}{2},n-\frac{1}{2}}\boldsymbol{U}^{t+\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}})]\frac{\Delta t}{2} + \boldsymbol{U}^t_{m,n} \end{aligned} \tag{5}$$

According to the positions of the equivalent step length in the equation (4) and (5), the calculation methods of equivalent step length are as follow

$$\begin{cases} {}^1\Delta r_{m,n} = S^e_{m,n} / (z_{m+\frac{1}{2},n-\frac{1}{2}} - z_{m-\frac{1}{2},n+\frac{1}{2}}) \\ {}^1\Delta z_{m,n} = S^e_{m,n} / (r_{m+\frac{1}{2},n-\frac{1}{2}} - r_{m-\frac{1}{2},n+\frac{1}{2}}) \\ {}^2\Delta r_{m,n} = S^e_{m,n} / (z_{m+\frac{1}{2},n+\frac{1}{2}} - z_{m-\frac{1}{2},n-\frac{1}{2}}) \\ {}^2\Delta z_{m,n} = S^e_{m,n} / (r_{m+\frac{1}{2},n+\frac{1}{2}} - r_{m-\frac{1}{2},n-\frac{1}{2}}) \end{cases} \tag{6}$$

$$\begin{cases} {}^1\Delta r_{m+\frac{1}{2},n+\frac{1}{2}} = S_{m+\frac{1}{2},n+\frac{1}{2}} / (z_{m+1,n} - z_{m,n+1}) \\ {}^1\Delta z_{m+\frac{1}{2},n+\frac{1}{2}} = S_{m+\frac{1}{2},n+\frac{1}{2}} / (r_{m+1,\ n} - r_{m,n+1}) \\ {}^2\Delta r_{m+\frac{1}{2},n+\frac{1}{2}} = S_{m+\frac{1}{2},n+\frac{1}{2}} / (z_{m+1,n+1} - z_{m,n}) \\ {}^2\Delta z_{m+\frac{1}{2},n+\frac{1}{2}} = S_{m+\frac{1}{2},n+\frac{1}{2}} / (r_{m+1,\ n+1} - r_{m,n}) \end{cases} \tag{7}$$

In order to ensure the numerical scheme stability, the time step Δt need to meet the condition is as follow

$$\Delta t \le \frac{\Delta l}{c_l} \quad (8)$$

Where, Δl is the smallest equivalent step length.

4 Example of the numerical method

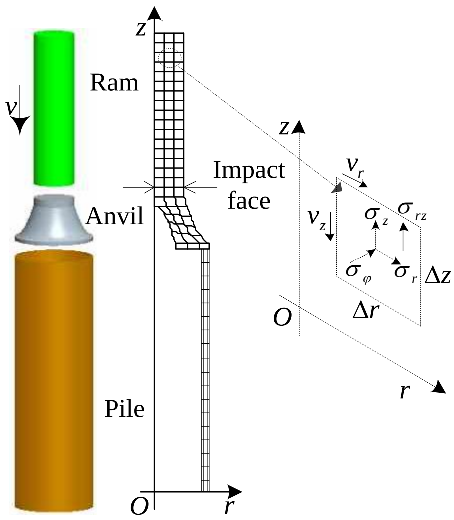

Figure 3. Axisymmetric geometric model of impact system.

The impact system of underwater pile driving hammer is composed of ram, anvil and pile, as shown in Figure 3. The dynamic response of each component can be described by the elastic dynamic equations under the cylindrical coordinate. The cross-section of ram and pile is rectangle. The dynamic response of ram and pile can be solved by Lax-wendorff finite difference scheme. Due to the cross-section of anvil contain curve, the dynamic response of anvil can be solved by the numerical method that present in section 2.

Parameters of the impact system are as follows in Table 1. The grid step length of ram and pile both are 1cm. The node numbers of the impact system is 82830. The time step length is 4.6ms.

Table 1. Parameters of the impact system.

Parameters	Values
Length of ram / m	6
Diameter of ram / m	1.3
Length of pile / m	30
Diameter of pile / m	2.5
Wall thickness of pile / m	0.04
Modulus of elasticity / GPa	207
Desity / kg/m3	7850
Impact velocity / m/s	6.26
Simulation time / ms	5.6

Figure 4 is simulation result of axial stress response of anvil, when the stress wave traveling through the anvil. The numerical result is convergence, and the wave front of the wave stress is remained.

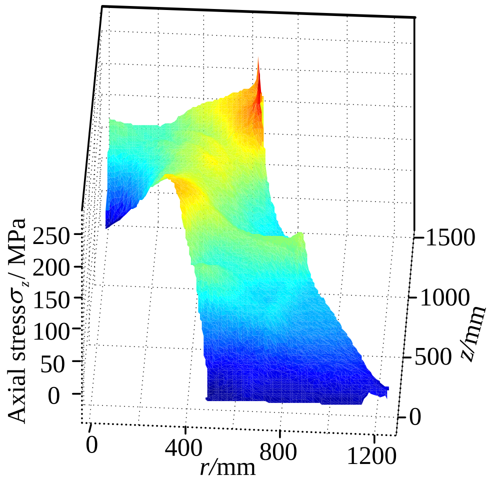

Figure 4. Axial stress response of anvil.

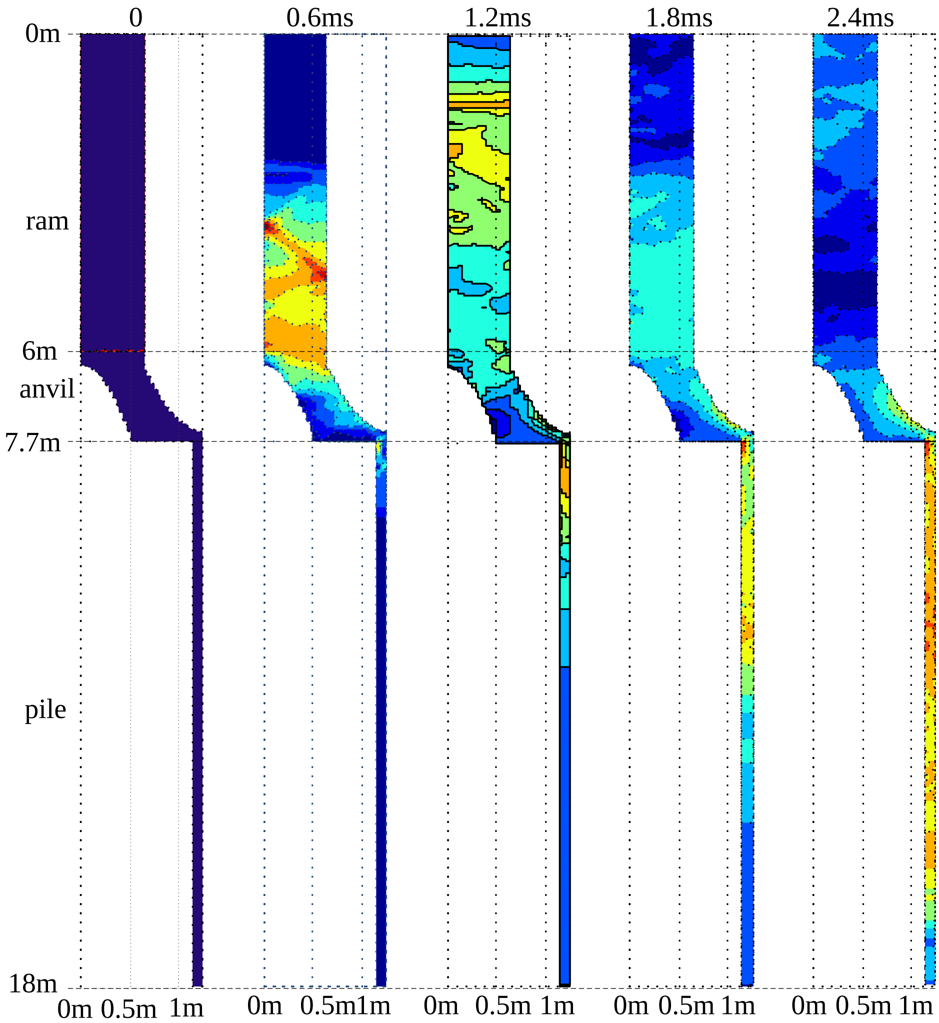

Figure 5. The changes of axial stress in impact system over time in one time collision.

Figure 5 shows the stress wave propagation process of impact system in one time collision. The axial stress of the impact system is zero when the collision starts. After 0.6ms, the stress wave propagates to the middle of the ram, and the stress wave propagates from the collision face of the anvil to the top of the pile. The anvil with curve outline delay the stress wave propagate to pile, so the distance of stress wave propagation in the pile is shorter than in the ram. After 1.2ms, the stress wave propagates to the free end of the ram, and the tensile stress wave start propagation from the free end of the ram to the collision end. After 1.8ms, the tensile stress wave propagates to the middle of the ram, and the area of ram have rebound speed which after the wave front of the tensile stress wave. After 2.4ms, the tensile stress wave propagate to the collision end of the ram, and the ram start to rebound, then the collision between ram and anvil is over.

According to Figure 5, the collision duration of the impact system is 2.4ms. By the one-dimensional characteristic method, the collision duration of the impact system is 2.34ms. The difference between the two results is because of the cushion of anvil. The simulation result can be reasonable explained by the theory of stress wave, and the simulation result conformed to the rule of stress wave. So the feasibility of the numerical method is proved which present in section 2.

5 Conclusions

Base on the Lax-Wendroff finite difference method, a new numerical method is presented for the dynamic response of axisymmetric object with curve in its cross-section. The collision simulation of the impact system of underwater pile driving hammer is realized, and the reasonable simulation results are obtained.

(1) The numerical scheme of axisymmetric elastodynamic equation under curve grid is derived.

(2) The equivalent step length of the element and the node is presented.

(3) The simulation result of impact system of underwater pile driving hammer conforms to the theory of stress wave.

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