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Specificity for interval-valued fuzzy sets

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Abstract

In this paper some axiomatic definitions about specificity for interval-valued fuzzy sets are proposed. Some examples of measures of specificity for interval-valued fuzzy sets are showed. It is also defined a extension of the notion of alpha cut for interval-valued fuzzy sets and a generalized similarity for interval-valued fuzzy relations. An axiomatic definition of specificity of interval-valued fuzzy sets under the knowledge of a generalized similarity is given.

Keywords: Specificity measure, Interval-valued fuzzy set, Similarity, T-indistinguishability.

1. Introduction

Interval-valued fuzzy sets ($\mathscr{IVFS}$) were introduced in the 60s by Grattan-Guinness [7], Jahn [8], Sambuc [9] and Zadeh [16]. They are extensions of classical fuzzy sets where the membership degree of the elements on the universe of discourse (between 0 and 1) is replaced by an interval in $[0,1]\times[0,1]$. They easily allow to model uncertainty and vagueness generalizing the fuzzy sets. Sometimes it is easier for experts to give a "membership interval" than a membership degree to a characteristic of objects on a universe. $\mathscr{IVFS}$ are a special case of type-2 fuzzy sets that simplifies the calculations while preserving their richness as well.

The concept of specificity provides a measure of the amount of information contained in a fuzzy set. It is strongly related to the inverse of the cardinality of a set. Specificity measures were introduced by Yager [10,11] showing its usefulness as a measure of tranquility when making a decision. The output information of expert systems and other knowledge-based system should be both specific and correct to be useful.

Measures of specificity have been widely analyzed [3,4,5], for intuitionistic fuzzy sets [14], for interval-valued fuzzy sets and for type 2 fuzzy sets [13].

2. Preliminaries

Let $X=\{e_1,...,e_n\}$ be a finite set.

Definition 2.1 *A fuzzy set μ on X is normal if there exists an element $x\in X$ such that $\mu(x)=1$.*

Definition 2.2 [11] *Let a_j be the j^{th} greatest membership degree of μ. A measure of specificity is a func-*

tion Sp:$\{a_j\} \to [0,1]$ *such that:*

- $Sp(\mu) = 1$ *if and only if* μ *is a singleton.*
- $Sp(\emptyset) = 0$
- $Sp(\mu)$ *depends on* a_j *in that way:*
  1. $\frac{\partial Sp(\mu)}{\partial a_1} > 0$
  2. $\frac{\partial Sp(\mu)}{\partial a_j} \leqslant 0$ *for all* $j \geqslant 2$

It is also defined a weaker measure of specificity:

Definition 2.3 [11] *Let* $[0,1]^X$ *be the class of fuzzy sets of X. A weak measure of specificity is a function Sp:*$[0,1]^X \to [0,1]$ *such that:*

- $Sp(\mu) = 1$ *if and only if* μ *is a singleton.*
- $Sp(\emptyset) = 0$
- *If* μ *and* η *are normal fuzzy sets in X and* $\mu \subset \eta$*, then* $Sp(\mu) \geqslant Sp(\eta)$*.*

Definition 2.4 *Let Sp and Sp′ be two measures of specificity. Sp is more strict than Sp′, denoted by* $Sp \leqslant Sp'$*, if for all sets,* μ*, it verifies:* $Sp(\mu) \leqslant Sp'(\mu)$*.*

Yager introduced [11] the linear measure of specificity on a finite space X as:

$$Sp_{\overrightarrow{w}}(\mu) = a_1 - \sum_{j=2}^{n} w_j a_j$$

where a_j is the j^{th} greatest membership degree of μ and $\{w_j\}$ is a set of weights verifying:

- $w_j \in [0,1]$
- $\Sigma_{j=2}^{n} w_j = 1$
- $\{w_j\}$ is not increasing.

Definition 2.5 [15] *A fuzzy relation* $R : X^2 \to [0,1]$ *is a similarity relation if it is reflexive, symmetric and transitive under the t-norm minimum* $(Min(R(a,b),R(b,c)) \leqslant R(a,c)$ *for all* a,b,c *in X).*

Yager also a defines a measure of specificity under the knowledge of a similarity to solve the Yager's jacket problem [12].

Definition 2.6 [12] *Let* μ *be a fuzzy set on X and let S be a similarity* $S : X \times X \to [0,1]$*. Let* π_α *be the set of classes of equivalence of the* α*-cut of S. The set of classes of equivalence under the knowledge of S* μ_α/S *is the subset of equivalence classes of the* α*-cut of S defined in that way: a equivalence class of the* α*-cut of S belongs to* μ_α/S *if its intersection with the* α*-cut of* μ_α *is not empty.*

Definition 2.7 [12] *Let* $[0,1]^X$ *be the set of fuzzy sets on X. Let* μ *be a fuzzy set on X and let S be a similarity* $S : X \times X \to [0,1]$*. The specificity of* μ *under S is defined as follows:*

$$Sp(\mu/S) = \int_0^{\alpha_{max}} \frac{1}{card(\mu_\alpha/S)} d\alpha$$

Definition 2.8 [2] *It is denoted by L and* $\leqslant_L$ *the following set and an order relation:*

1. $L = \{[x_1,x_2] \in [0,1]^2$ *with* $x_1 \leqslant x_2\}$*.*
2. $[x_1,x_2] \leqslant_L [y_1,y_2]$ *if and only if* $x_1 \leqslant y_1$ *and* $x_2 \leqslant y_2$

Also by definition:

$[x_1,x_2] <_L [y_1,y_2] \Leftrightarrow x_1 < y_1, x_2 \leqslant y_2$ or
$x_1 \leqslant y_1, x_2 < y_2$
$[x_1,x_2] =_L [y_1,y_2] \Leftrightarrow x_1 = y_1, x_2 = y_2$.

$0_L =_L [0,0]$ and $1_L =_L [1,1]$ are the smallest and the greatest elements in L respectively.

L is a complete lattice and the supremum and infimum are defined as follows:

Definition 2.9 [1] *Let* $\{[v_i,w_i]\}$ *be a set of intervals on L. Then*

1. $Meet\{[v_i,w_i]\} \equiv [infimun\{v_i\}, infimun\{w_i\}]$
2. $Joint\{[v_i,w_i]\} \equiv [supremun\{v_i\}, supremun\{w_i\}]$

Definition 2.10 [2] *An interval-valued fuzzy set A on a universe X can be represented by the mapping:*

$$A : X \to [0,1]^2$$

Definition 2.11 [2] *Let X be a universe and A and B two interval-valued fuzzy sets. The equality between A and B is defined as:* $A =_L B$ *if and only if* $A(a) =_L B(a)\ \forall a \in X$*.*

Definition 2.12 [2] *Let X be a universe and A and B two interval-valued fuzzy sets. The inclusion of A in to B is defined as:* $A \subseteq_L B$ *if and only if* $A(a) \leqslant_L B(a)$ $\forall a \in X$*.*

Definition 2.13 [2] *An interval-valued negation* $\mathcal{N}$ *is a decreasing function,* $\mathcal{N} : L \to L$*, that satisfies:*

1. $\mathcal{N}(0_L) =_L 1_L$
2. $\mathcal{N}(1_L) =_L 0_L$

If $\mathcal{N}(\mathcal{N}([x_1,x_2])) =_L [x_1,x_2]$ then $\mathcal{N}$ is called an involutive negation.

Definition 2.14 *A strong interval-valued negation $\mathcal{N}$ is a strictly decreasing and involutive function, $\mathcal{N} : L \to L$, that satisfies:*

1. $\mathcal{N}(0_L) =_L 1_L$
2. $\mathcal{N}(1_L) =_L 0_L$

Example 2.1 *Let $\mathcal{N}$ be the involutive mapping defined by:*

$$\mathcal{N} : L \to L$$
$$\mathcal{N}([x_1,x_2]) =_L [1-x_2, 1-x_1]$$

Then $\mathcal{N}$ is a negation operator for interval-valued fuzzy sets. It is trivial to prove that: $\mathcal{N}(0_L) =_L 1_L$, $\mathcal{N}(1_L) =_L 0_L$ and $\mathcal{N}(\mathcal{N}([x_1,x_2])) =_L [x_1,x_2]$.

Definition 2.15 [2] *A generalized t-norm function $\mathcal{T}$ is a monotone increasing, symmetric and associative operator, $\mathcal{T} : L^2 \to L$, that satisfies: $\mathcal{T}(1_L,[x_1,x_2]) =_L [x_1,x_2]$ for all $[x_1,x_2]$ in L.*

Example 2.2 *Let Inf_L be defined as follows:*

$Inf_L([x_1,x_2],[y_1,y_2]) = Meet\{[x_1,x_2],[y_1,y_2]\}$

It easy to prove that Inf_L is a generalized t-norm.

3. Specificity for Interval-valued Fuzzy Sets

Definition 3.1 *A operator $G : [0,1]^n \to [0,1]$ is an operator of specificity if it is continuous and it is increasing for the first argument and decreasing for the others and satisfies:*

- $G(1,0...0) = 1$
- $G(0,0...0) = 0$

Lemma 3.1 *Let μ be a fuzzy set on X. Let $\{\mu(a_i)\}$ for all $i = 1..n$ the list of membership degrees of μ decreasing order. Let $G : [0,1]^n \to [0,1]$ be an operator of specificity. Then $G(\mu(a_1),...,\mu(a_n))$ is a measure of specificity for $\mathcal{FS}$s.*

Proof. trivial by definition 2.2 □

Definition 3.2 *An operator $f(x,y) : [0,1]^2 \to [0,1]$ with $x \leqslant y$ is called transformation operator if it is continuous, increasing and verifies:*

1. $f(1,1) = 1$
2. $f(0,0) = 0$
3. $f(0,x) > 0$ *for all* $x \in (0,1]$
4. $f(x,1) < 1$ *for all* $x \in [0,1)$

Some examples of transformation operators are the following:

Example 3.1

$$f(x,y) = \frac{x+y}{2}$$

Example 3.2

$$f(x,y) = \alpha * x + \beta * y$$

with $\alpha + \beta = 1, \alpha > 0, \beta > 0$

Example 3.3

$$f(x,y) = \frac{x^2+y^2}{2}$$

Definition 3.3 *Let μ be an interval-valued fuzzy set on X and let $\{[x_{1_q},x_{2_q}]\}$ for all $q : 1..n$ be its membership intervals. Let f be a transformation operator. Then, the f-list of μ is the set of all the membership intervals of elements of X, ordered decreasingly through the operator f, that is, $[x,y] \leqslant_f [z,t]$ if and only if $f(x,y) \leqslant f(z,t)$.*

Example 3.4 *Let X be the universe with cardinality 5 and let μ be the following interval-valued fuzzy set:*

$$\mu = \{[0.8,0.9]/e_1, [0.2,0.4]/e_2, [0.8,1.0]/e_3, [0.1,0.2]/e_4, [0.0,0.1]/e_5\}$$

Then, if $f(x,y) = (x+y)/2$ then:

[x,y]	*f(x,y)*
[0.8,0.9]	*0.85*
[0.2,0.4]	*0.30*
[0.8,1.0]	*0.90*
[0.1,0.2]	*0.15*
[0.0,0.1]	*0.05*

The f-list of μ is:

$\{[0.8,1.0],[0.8,0.9],[0.2,0.4],[0.1,0.2],[0.0,0.1]\}$

Definition 3.4 *An interval-valued fuzzy set μ on X is a singleton if there exists an element $a_i \in X$ such that $\mu(a_i) = 1_L$ and $\mu(a_j) = 0_L$ (for all $j \neq i$) for the others.*

Definition 3.5 *Let $([0,1]^2)^X$ be the set of interval-valued fuzzy sets on X. Let f be a transformation operator. Let $\{[x_{1_q}, x_{2_q}]\}$ for all $q = 1..n$ be the f-list of μ. A f-measure of specificity for interval-valued fuzzy sets is a function $Sp_f : ([0,1]^2)^X \rightarrow [0,1]$ such that:*

- *$Sp_f(\mu) = 1$ if and only if μ is a singleton.*
- *$Sp_f(\emptyset) = 0$.*
- *If $[x_{1_1}, x_{2_1}]$ increases (according to $\leqslant_L$) then $Sp_f(\mu)$ increases.*
- *If $[x_{1_q}, x_{2_q}]$ increases (according to $\leqslant_L$) then $Sp_f(\mu)$ decreases for all $q : 2..n$.*

Definition 3.6 *An interval-valued fuzzy set μ on X is normal if there exists an element $a \in X$ such that $\mu(a) = 1_L$.*

Definition 3.7 [6] *Let $([0,1]^2)^X$ be the set of membership degrees of interval-valued fuzzy sets on X. A weak measure of specificity for interval-valued fuzzy sets is a function Sp:$([0,1]^2)^X \rightarrow [0,1]$ such that:*

- *$Sp(\mu) = 1$ if and only if μ is a singleton.*
- *$Sp(\emptyset) = 0$*
- *If μ and η are normal fuzzy sets in X and $\mu \subseteq_L \eta$, then $Sp(\mu) \geqslant Sp(\eta)$.*

Lemma 3.2 *If Sp_f is an f-measure of specificity for interval-valued fuzzy sets then Sp_f is a weak measure of specificity for interval-valued fuzzy sets.*

Proof. Let $\{[x_{1_q}, x_{2_q}]\}$ and $\{[y_{1_q}, y_{2_q}]\}$ for all $q = 1..n$ be the f-list of μ and η respectively. If μ and η are normal and $\mu \subseteq_L \eta$ then $[x_{1_q}, x_{2_q}] \leqslant_L [y_{1_q}, y_{2_q}]$ for all $q = 2..n$. According to the fourth axiom of the definition 3.5 $Sp_f(\mu) \geqslant Sp_f(\eta)$ □

Example 3.5 *In [13] Yager shows a particular case of function of transformation, f, (called Q_F). Let μ be an interval-valued fuzzy set on X with $\mu(a_q) = [x_{1_q}, x_{2_q}]$ for all $q : 1..n$.*

$Q_F(a_i) = f(x_{1_q}, x_{2_q})$ such that $x \leqslant f(x,y) \leqslant y$ for all x,y.

Let a_i be the element of X which maximizes Q_F. Then, the following expression is a measure of specificity for interval-valued fuzzy sets:

$$Sp = Q_F(a_i) - \frac{1}{n-1}\sum_{\forall k \neq i} Q_F(a_k).$$

Lemma 3.3 *Let μ be an interval-valued fuzzy set on X and let Sp_f be any f-measure of specificity over μ. Let $\{[x_{1_q}, x_{2_q}]\}$ for all $q : 1..n$ the f-list of μ. Then, there exists an operator of specificity $G : [0,1]^n \rightarrow [0,1]$ such that:*

$$Sp_f(\mu) = G(f(x_{1_1}, x_{2_1}), ..., f(x_{1_n}, x_{2_n})) \quad (1)$$

Corollary 3.1 *Let G be a measure of specificity for $\mathcal{FS}$s. Let f a transformation operator. Then $G(f(x_{1_1}, x_{2_1}), ..., f(x_{1_n}, x_{2_n}))$ is a f-measure for $\mathcal{IVFS}$s*

Definition 3.8 *Let Sp_f and Sp'_g be two measures of specificity. Sp_f is more strict than Sp'_g, denoted by $Sp_f \leqslant Sp'_g$, if for all set, μ, it verifies: $Sp_f(\mu) \leqslant Sp'_g(\mu)$.*

Theorem 3.1 *Sp_f is more strict than Sp'_g if and only if $f(x,y) \leqslant g(x,y)$ for all x, y.*

Proof. Trivial □

Theorem 3.2 *Let f be a transformation operator and $\{\alpha_i\}$ a set of weights that satisfies:*

- *$\alpha_j \in (0,1]$*
- *$\Sigma_{j=2}^{n} \alpha_j = 1$*
- *$\{\alpha_j\}$ is not increasing.*

Let T, T′, S and N be, two t-norms, a t-conorm and a negation (in [0,1],$\leqslant$) respectively. Let $\{f(x_{1_k}, x_{2_k})\}$ be the f-list of an interval-valued fuzzy set μ. Then

$$Sp_f(\mu) = T(f(x_{1_1}, x_{2_1})_1, N(S(T'(\alpha_2, f(x_{1_2}, x_{2_2}))), ...$$
$$..., T(\alpha_n, f(x_{1_n}, x_{2_n})))))$$

is a f-measure of specificity for interval-valued fuzzy set.

This expression is a generalization of the t-norm based measure of specificity given in [3] but extended for $\mathcal{IVFS}$.

Proof.

1. $Sp_f(\mu) = 1$ if and only if μ is a singleton:
   - If μ is a singleton then $[x_{1_1}, x_{2_1}] = [1,1]$ and $[x_{1_k}, x_{2_k}] = [0,0]$ for all $k > 1$. Then $f(x_{1_1}, x_{2_1}) = 1$ and $f(x_{1_k}, x_{2_k}) = 0$ for all $k > 1$.

- If $Sp_f(\mu) = 1$, it is necessary that $f(x_{1_1}, x_{2_1}) = 1$ and

  $$S(T(\alpha_2, f(x_{1_2}, x_{2_2}))), \ldots, T(\alpha_n, f(x_{1_n}, x_{2_n})) = 0$$

  Then $T(\alpha_k, f(x_{1_k}, x_{2_k}))) = 0$ for all k and $f(x_{1_k}, x_{2_k}) = 0$ for all k.

2. $Sp_f(\emptyset) = 0$: trivial.

3. Trivial due to the fact T, T' and S are monotonic

□

Let $\{\alpha_i\}$ be a set of weights which satisfies the conditions of theorem 3.2.

Example 3.6 *With $T(a,b) = Max\{0, a+b-1\}$, $N(a) = 1-a$, $S(a,b) = Min\{1, a+b\}$, $T'(a,b) = a*b$ and $f(x,y) = \frac{x+y}{2}$, it is obtained:*

$$Sp_f(\mu) = \frac{1}{2}(x_{1_1} + x_{2_1}) - \sum_{j=2}^{n} \alpha_j (x_{1_j} + x_{2_j})$$

Example 3.7 *With $T(a,b) = Max\{0, a+b-1\}$, $N(a) = 1-a$, $S(a,b) = Min\{1, a+b\}$, $T'(a,b) = a*b$ and $f(x,y) = \alpha * x + \beta * y$ with $\alpha + \beta = 1, \alpha > 0, \beta > 0$, it is obtained:*

$$Sp_f(\mu) = \alpha * x_{1_1} + \beta * x_{2_1} - \sum_{j=2}^{n} \alpha_j (\alpha * x_{1_j} + \beta * x_{2_j})$$

Example 3.8 *With $T(a,b) = Max\{0, a+b-1\}$, $N(a) = 1-a$, $S(a,b) = Min\{1, a+b\}$, $T'(a,b) = a*b$ and $f(x,y) = \frac{x^2+y^2}{2}$, it is obtained:*

$$Sp_f(\mu) = \frac{1}{2}(x_{1_1}^2 + x_{2_1}^2) - \frac{1}{2} \sum_{j=2}^{n} \alpha_j * (x_{1_j}^2 + x_{2_j}^2)$$

Examples 3.6 and 3.7 are extensions of R. Yager's linear measure of specificity [11] for $\mathscr{IVFS}$.

4. Alpha cuts for interval-valued fuzzy sets

Definition 4.1 *Let μ be an interval-valued fuzzy set on X. The α_1, α_2 cuts of μ are subsets of X defined as follows:*

$$\mu_{\alpha_1,\alpha_2} = \{a_i \mid \mu(a_i) \geqslant_L [\alpha_1, \alpha_2]\}$$

Definition 4.2 *Let R be an interval-valued relation $R : X^2 \to L$. The α_1, α_2 cut of R, R_{α_1,α_2}, is a crisp relation defined for all α_1, α_2 in [0,1] as follows:*

$$R_{\alpha_1,\alpha_2}(a_i, a_j) = \begin{cases} 1 & R(a_i, a_j) \geqslant_L [\alpha_1, \alpha_2]; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.1 *Let $R = [R_{down}, R_{up}]$ be an interval-valued fuzzy relation on X where R_{down} and R_{up} are fuzzy relations on X, it is, $R(a_i, a_j) = [R_{down}(a_i, a_j), R_{up}(a_i, a_j)]$ for all a_i, a_j in X. Then, $R_{\alpha_1,\alpha_2}(a_i, a_j) = 1$ if and only if $R_{down\ \alpha_1}(a_i, a_j) = 1$ and $R_{up\ \alpha_2}(a_i, a_j) = 1$*

Proof. Trivial due to definition 4.2 □

Lemma 4.2 *Let R, S be two fuzzy relations. If $R_\alpha(a_i, a_j) = S_\alpha(a_i, a_j)$ for all a_i, a_j on X and for all α in $[0,1]$ then $R(a_i, a_j) = S(a_i, a_j)$.*

Proof. Let's suppose that there exist r, s such that: $R(a_r, a_s) \neq S(a_r, a_s)$. If $R(a_r, a_s) > S(a_r, a_s)$ then $R_p(a_r, a_s) = 1$ and $S_p(a_r, a_s) = 0$ where $p = R(a_r, a_s)$ which is a contradiction. If $R(a_r, a_s) < S(a_r, a_s)$ a similar contradiction can be found. □

Proposition 4.1 *The set of all α_1, α_2 cuts of an interval-valued fuzzy relation R determine R.*

Proof. By lemma 4.1 the α_1, α_2 cuts of an interval-valued fuzzy relation R are determined by the α cuts of R_{down} and the α cuts of R_{up}, which by lemma 4.2 are determined by the fuzzy relations R_{down} and R_{up}, that define $R = [R_{down}, R_{up}]$, so the α_1, α_2 cuts of R determine R. □

Corollary 4.1 *Let R, S be two interval-valued fuzzy relations. If $R_{\alpha_1,\alpha_2}(a_i, a_j) = S_{\alpha_1,\alpha_2}(a_i, a_j)$ for all a_i, a_j on X and for all α_1, α_2 in $[0,1]$ then $R(a_i, a_j) = S(a_i, a_j)$.*

Proof. Trivial due to proposition 4.1 □

Definition 4.3 *Let $\mathscr{T}$ be a generalized t-norm [2]. An interval-valued relation $R : X^2 \to L$ is a generalized $\mathscr{T}$-indistinguishability if it is reflexive, symmetric and $\mathscr{T}$-transitive, it is:*

1. *$R(a,a) =_L 1_L$ for all a in X.*

2. *$R(a,b) =_L R(b,a)$ for all a, b in X.*

3. *$\mathscr{T}(R(a,b),R(b,c)) \leqslant_L R(a,c)$ for all a,b,c in X.*

Lemma 4.3 *Let $R : X^2 \to L$ be a generalized Inf_L-indistinguishability. Then, for each α_1, α_2, R_{α_1,α_2} is an equivalence relation.*

Proof.

1. $R_{\alpha_1,\alpha_2}(a_i,a_i) = 1$ trivially.
2. $R_{\alpha_1,\alpha_2}(a_i,a_j) = R_{\alpha_1,\alpha_2}(a_j,a_i)$ trivially.
3. Due to the fact that R is a Inf_L-indistinguishability:

$$Inf_L(R(a_i,a_k),R(a_k,a_j)) \leqslant_L R(a_i,a_j) \text{ for all } a_i,a_j,a_k \text{ in } X$$

If $R_{\alpha_1,\alpha_2}(a_i,a_k) = 1$ and $R_{\alpha_1,\alpha_2}(a_k,a_j) = 1$ then $R(a_i,a_k) \geqslant_L [\alpha_1,\alpha_2]$ and $R(a_k,a_j) \geqslant_L [\alpha_1,\alpha_2]$ and

$$[\alpha_1,\alpha_2] \leqslant_L Inf_L(R(a_i,a_k),R(a_k,a_j)) \leqslant_L R(a_i,a_j)$$

therefore: $[\alpha_1,\alpha_2] \leqslant_L R(a_i,a_j)$ and so R_{α_1,α_2} is transitive

□

Lemma 4.4 *Let $R : X^2 \to L$ be an interval-valued relation. If for each α_1, α_2, R_{α_1,α_2} is an equivalence relation, then R is a Inf_L-indistinguishability.*

Proof.

1. $R(a_i,a_j) = 1$ by contradiction.
2. $R(a_i,a_j) = R(a_j,a_i)$ by contradiction.
3. It is supposed that R is not a Inf_L-indistinguishability:

$$Inf_L(R(a_i,a_k),R(a_k,a_j)) >_L R(a_i,a_j) \text{ for some } a_i,a_j,a_k \text{ in } X$$

Then, it is found a R_{α_1,α_2} that is not a equivalence relation: Let ε and δ be two real number arbitrarily small such that $\alpha_1 = \underline{R(a_i,a_j)} - \varepsilon$ and $\alpha_2 = \overline{R(a_i,a_j)} - \delta$. Then $R_{\alpha_1,\alpha_2}(a_i,a_k) = 1$ and $R_{\alpha_1,\alpha_2}(a_k,a_j) = 1$ but $R_{\alpha_1,\alpha_2}(a_i,a_j) = 0$, i.e R_{α_1,α_2} is not transitive

□

Theorem 4.1 *Let $R : X^2 \to L$ be an interval-valued relation. If for each α_1, α_2, R_{α_1,α_2} is an equivalence relation if and only if R is a Inf_L-indistinguishability.*

Proof. Trivial due to the lemmas 4.3 and 4.4 □

Corollary 4.2 *Let $R : X^2 \to L$ be an interval-valued relation. Then, R is a Inf_L-indistinguishability if and only if $\underline{R_{\alpha_1,\alpha_2}}$ and $\overline{R_{\alpha_1,\alpha_2}}$ are equivalence relations for all α_1, α_2.*

Theorem 4.2 *Let $R : X^2 \to L$ be a generalized $\mathscr{T}$-indistinguishability (with $\mathscr{T} \neq Inf_L$). Then, there exists some α_1, α_2, such that R_{α_1,α_2} is not an equivalence relation.*

Proof. Let $R : X^2 \to L$ be a generalized $\mathscr{T}$-indistinguishability (with $\mathscr{T} \neq Inf_L$). Let a_i, a_j, a_k be elements of the universe X such that: $\mathscr{T}(R(a_i,a_k),R(a_k,a_j)) =_L R(a_i,a_j)$. Let $[\alpha_1,\alpha_2]$ be such that: $[\alpha_1,\alpha_2] = Inf_L R(a_i,a_k),R(a_k,a_j)$. Then, due the fact that Inf_L is the greatest of the generalized t-norms :

$$R(a_i,a_j) = \mathscr{T}(R(a_i,a_k),R(a_k,a_j)) \leqslant_L Inf_L R(a_i,a_k),R(a_k,a_j)$$

Then $R_{\alpha_1,\alpha_2}(a_i,a_k) = 1$, $R_{\alpha_1,\alpha_2}(a_k,a_j) = 1$ but $R_{\alpha_1,\alpha_2}(a_i,a_j) = 0$ □

5. Specificity for Interval-valued Fuzzy Sets under generalized similarities

Proposition 5.1 *Let μ be an interval-valued fuzzy set on X. Let $[\widehat{\alpha_1}, \widehat{\alpha_2}] = Joint\{\mu(a_i)\}$ for all i:1..n. Then:*

$$Sp(\mu) = 2 * \int_0^{\widehat{\alpha_2}} \int_0^{\alpha_2} \frac{1}{card(\mu_{\alpha_1,\alpha_2})} d\alpha_1\, d\alpha_2 + \int_{\widehat{\alpha_1}}^{\widehat{\alpha_2}} \int_0^{\widehat{\alpha_1}} \frac{1}{card(\mu_{\alpha_1,\alpha_2})} d\alpha_1\, d\alpha_2$$

It is a measure of specificity for $\mathscr{IVFS}$s.

Note that the integration area guarantees that $card(\mu_{\alpha_1,\alpha_2})$ is not zero.

Proof.

- Axiom 1:
  1. If μ is a singleton then $Sp(\mu/S) = 1$:

- Let a_k be the only element on X such that $\mu(a_k) = 1_L$.
- Then $\mu_{\alpha_1,\alpha_2} = a_k$ for all α_1, α_2 and $card(\mu_{\alpha_1,\alpha_2}) = 1$ for all α_1, α_2 and $[\widehat{\alpha_1}, \widehat{\alpha_2}] = [1,1]$.
- Then
$$2 * \int_0^1 \int_0^{\alpha_2} 1 d\alpha_1\, d\alpha_2 = 1$$

2. So that $Sp(\mu) = 1$ it is necessary that $[\widehat{\alpha_1}, \widehat{\alpha_2}] = [1,1]$ and $card(\mu_{\alpha_1,\alpha_2}) = 1$. Otherwise $Sp(\mu) < 1$. Hence μ is a singleton.

- Axiom 2:
Trivial.
- Axiom 3: Let $\{[x_{1_q}, x_{2_q}]\}$ for all $q = 1..n$ be the f-list of μ.
  1. If $[x_{1_1}, x_{2_1}]$ increases then $[\widehat{\alpha_1}, \widehat{\alpha_2}]$ increases and $card(\mu_{\alpha_1,\alpha_2})$ does not change.
  2. If $[x_{1_q}, x_{2_q}]$ for all $q : 2..n$ increases then $1/card(\mu_{\alpha_1,\alpha_2})$ decreases

□

In [4] a set of axioms that generalize the specificity of a fuzzy set under T-indistinguishabilities is given.

Definition 5.1 [4] *Let Sp a measure of specificity for* $\mathscr{IVFS}$*s.* $Sp(\mu/S)$ *is a measure of specificity under a generalized similarity S if it verifies:*

1. $Sp(\mu/S) = 1$ *if and only if* μ *is a singleton.*
2. $Sp(\emptyset/S) = 0$.
3. $Sp(\mu/Id) = Sp(\mu)$.
4. $Sp(\mu/S) \geqslant Sp(\mu)$.

Definition 5.2 *An interval-valued relation* $R : X^2 \to L$ *is a generalized similarity if it is reflexive, symmetric and* Inf_L*-transitive where* $Inf_L([x_1,x_2],[y_1,y_2]) = [min(x_1,y_1), min(x_2,y_2)]$*, it is, R is an* Inf_L*-indistinguishability.*

Definition 5.3 *Let* μ *be a fuzzy set on X and let S be a similarity* $S : X \times X \to [0,1]$*. Let* π_{α_1,α_2} *be the set of classes of equivalence of the* α_1, α_2 *cut of S. The set of classes of equivalence under the knowledge of S* $\mu_{\alpha_1,\alpha_2}/S$ *is the subset of equivalence classes of the* α_1, α_2 *cut of S defined in that way: a equivalence class of the* α_1, α_2 *cut of S belongs to* $\mu_{\alpha_1,\alpha_2}/S$ *if its intersection with* μ_{α_1,α_2} *is not empty.*

Example 5.1 *Let* $E = \{e_1, e_2, e_3, e_4\}$*. Let* $\mu = \{[0.6,0.8]/e_1 + [0.7,0.8]/e_2 + [0.8,0.8]/e_3 + [0.9,1.0]/e_4\}$ *and*

$$S = \begin{pmatrix} 1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 1 & 0.8 & 0.6 \\ 0.1 & 0.8 & 1 & 0.6 \\ 0.1 & 0.6 & 0.6 & 1 \end{pmatrix}$$

$$R_{0.7,0.8} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, $\pi_{0.7,0.8} = \{\{e_1\},\{e_2,e_3\},\{e_4\}\}$ $\mu_{0.7,0.8} = \{e_2, e_3, e_4\}$ *and* $\pi_{0.7,0.8}/S = \{\{e_2,e_3\},\{e_4\}\}$

Proposition 5.2 *Let* μ *be an interval-valued fuzzy set on X and let S be a similarity* $S : X \times X \to [0,1]$*. Then:*

$$Sp(\mu/S) = 2 * \int_0^{\widehat{\alpha_2}} \int_0^{\alpha_2} \frac{1}{card(\mu_{\alpha_1,\alpha_2/S})} d\alpha_1\, d\alpha_2 + \int_{\widehat{\alpha_1}}^{\widehat{\alpha_2}} \int_0^{\widehat{\alpha_1}} \frac{1}{card(\mu_{\alpha_1,\alpha_2/S})} d\alpha_1\, d\alpha_2$$

It is a measure of specificity for $\mathscr{IVFS}$*s.*

Note that the integration area guarantees that $card(\mu_{\alpha_1,\alpha_2/S})$ is not zero.

Proof. Let $\{\pi^i_{\alpha_1,\alpha_2}\}$ for all i be the set of equivalence classes of π_{α_1,α_2}.

- Axiom 1:
  1. If μ is a singleton then $Sp(\mu/S) = 1$:
     - Let a_k be the only element on X such that $\mu(a_k) = 1_L$.
     - Then $\mu_{\alpha_1,\alpha_2} = a_k$ for all α_1, α_2.
     - There exists only a $\pi^i_{\alpha_1,\alpha_2}$ such that a_k belongs to it.
     - And $card(\mu_{\alpha_1,\alpha_2}/S) = 1$ for all α_1, α_2.
     - Then
     $$2 * \int_0^1 \int_0^{\alpha_2} 1 d\alpha_1\, d\alpha_2 = 1$$
  2. If $Sp(\mu/S) = 1$ then μ is a singleton:
  If $Sp(\mu/S) = 1$ then $card(\mu_{\alpha_1,\alpha_2}/S) = 1$ for all α_1, α_2 and μ is a singleton.
- Axiom 2:
Trivial.

- Axiom 3:
  Remember that $X = \{a_1, ..., a_n\}$, then if R is the relation identity then $\{\pi^i_{\alpha_1,\alpha_2}\} = a_i$ for all $i : 1..n$ and $card(\mu_{\alpha_1,\alpha_2}/S) = card(\mu_{\alpha_1,\alpha_2})$.
- Axiom 4:
  For a relation S there will exist α_1, α_2 such that $card(\pi^i_{\alpha_1,\alpha_2}) > 1$ and $card(\mu_{\alpha_1,\alpha_2}/S) < card(\mu_{\alpha_1,\alpha_2})$

□

6. Conclusion

Several expression for t-norm based measure of specificity for $\mathscr{IVFS}s$ have been proposed and studied.

An generalized expression for measures of specificity have been proposed for $\mathscr{IVFS}s$ and the measures of specificity under the knowledge of generalized similarities have also been defined following the Yager's jacket ideas.

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