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Group decision making methods of the incomplete IFPRs and IPRs

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Abstract

We propose optimal priority methods on the incomplete intuitionistic fuzzy preference relation (IFPR) and the incomplete interval preference relation (IPR). The least squares method has been used previously to derive the priority vector of the fuzzy preference relation (FPR). In this paper, we generalize the least squares method to IFPR and IPR based on our proposed multiplicative consistent conditions. We also investigate the relationships between the optimal models of incomplete IFPRs, IPRs and FPRs. We also apply the same method to the case of collective judgment with complete information. We illustrate the feasibility and effectiveness of our proposed methods with three numerical examples.

Keywords: Group decision-making; Preference relation; Least squares model; Priority

1. Introduction

In multiple attribute decision making, decision makers (DMs) provide their subjective opinions by comparing each pair of alternatives and then constructing judgment matrices [1,2] to order a finite number of alternatives from best to worst. Different DMs may have different preferences, and the judgment matrices (also called preference relations) may therefore take many forms. Examples include the fuzzy preference relation (FPR) [3,4] and the interval preference relation (IPR) [5]. In FPR, the elements denote the membership degree to which one alternative is preferred to another. They range between 0 and 1. This key idea originates from Zadeh's fuzzy sets [6]. However, in IPR, the elements denote the range of the membership degree to which one alternative is preferred to another. They characterized by a closed subinterval of $[0,1]$. This key idea comes from interval-valued fuzzy sets (IVFS) of Zadeh [7].

In 1986, Atanassov [8,9] generalized Zadeh's fuzzy sets to intuitionistic fuzzy sets (IFS). IFS are well suited to dealing with inevitably imprecise or not totally reliable judgment [10,11]. Szmidt and Kacprzyk [12,13] introduce an intuitionistic fuzzy preference relation (IFPR) to study the consensus-reaching process, and to analyze the extent of agreement within a group of experts. Xu [14,15,16,17] investigates the properties of IFPRs by constructing a score matrix and an accuracy matrix. He also researches the group decision method with IFPRs.

In practical decision making problems, because of either the uncertainty of objective things, or the vague nature of human beings, some of the preference degree values may not be presented by DMs. A preference relation with some entries missing is called an incomplete preference relation. Much research has been devoted to this situation. The earliest attempt to obtain the priorities of incomplete triangular fuzzy number preference relations using the logarithmic least squares method (LLSM) was by Laarhoven and Pedrycz [18]. Kwiesielewicz [19,20] generalized this work using a pseudo-inverse method. Xu [21] obtained the priority vector of the incomplete FPR by developing goal programming methods. Herrera-Viedma et al. [22,23,24] proposed a consensus model for group decision making with incomplete FPRs. Wei et al. [25,26,27] introduced novel induced aggregating operators with fuzzy number intuitionistic fuzzy information to group decision making. We now develop the priority approach on incomplete IFPRs and IPRs.

The paper is organized as follows. In Section 2, the definition of the IFPR and its consistent conditions are proposed, based on the multiplicative consistent definition of the IPR, respectively. In Section 3, we present the optimal priority models of incomplete IFPRs, IPRs and FPRs. We also apply the methods to the cases of collective judgment with complete information. In Section 4, we give numerical examples to illustrate the validity and practicality of the proposed methods. A short conclusion is given in Section 5.

2. Basic concepts

2.1. Three kinds of fuzzy sets

Let $X = \{x_1, \ldots, x_n\}$ be an ordinary finite non-empty set.

A fuzzy set [6] F in X is an expression given by $F = \{< x, \mu_F(x) > |x \in X\}$, where $\mu_F : X \mapsto [0,1]$ is the membership function of F, and $\mu_F(x) \in [0,1]$ deotes the degree of membership of $x \in X$ in F.

An interval-valued fuzzy set (IVFS) [7] I in X is an expression given by $I = \{< x, M_I(x) > |x \in X\}$, where $M_I : X \mapsto D[0,1]$ such that $M_I(x) = [M_{IL}(x), M_{IU}(x)]$, $D[0,1]$ being the set of all closed subintervals of $[0,1]$, $M_{IL}(x)$ and $M_{IU}(x)$ are the lower extreme and the upper extreme, respectively, of the interval $M_I(x)$. The IVFS is the extension of Zadeh's fuzzy set.

An intuitionistic fuzzy set (IFS) [8] in X is an expression given by $A' = \{< x, \mu_{A'}(x), \nu_{A'}(x) > |x \in X\}$, where $\mu_{A'} : X \mapsto [0,1]$, $\nu_{A'} : X \mapsto [0,1]$ with the condition $0 \leqslant \mu_{A'}(x) + \nu_{A'}(x) \leqslant 1$, for all x in X. The numbers $\mu_{A'}(x)$ and $\nu_{A'}(x)$ denote, respectively, the membership degree and the non-membership degree of the element x in A'.

For each finite intuitionistic fuzzy set in X, $\pi_{A'}(x) = 1 - \mu_{A'}(x) - \nu_{A'}(x)$ is called an intuitionistic fuzzy index of A'. It is a hesitation degree of whether x belongs to A' or not. It is obvious that $0 \leqslant \pi_{A'}(x) \leqslant 1$ for each $x \in A'$. If $\pi_{A'}(x) = 0$, then $\mu_{A'}(x) + \nu_{A'}(x) = 1$, which indicates that the intuitionistic fuzzy set A' has degenerated to the classic fuzzy set $A' = \{< x, \mu_{A'}(x) > |x \in X\}$.

IVFS and IFS are based on different semantics [28]. However, from a mathematical point of view, the elements of IVFS and the elements of IFS can be transformed each other [29,30,31]. Let $B' = \{< x, \mu_{B'}(x), \nu_{B'}(x) > |x \in X\}$ be an IFS, and $\pi_{B'}(x) = 1 - \mu_{B'}(x) - \nu_{B'}(x)$ be an intuitionistic fuzzy index of B'. If we combine $\mu_{B'}(x)$ with $\pi_{B'}(x)$, and combine $\nu_{B'}(x)$ with $\pi_{B'}(x)$, then we can get two intervals $B_1 = [\mu_{B'}(x), \mu_{B'}(x) + \pi_{B'}(x)] = [\mu_{B'}(x), 1 - \nu_{B'}(x)]$ and $B_2 = [\nu_{B'}(x), \nu_{B'}(x) + \pi_{B'}(x)] = [\nu_{B'}(x), 1 - \mu_{B'}(x)]$, respectively. Conversely, intervals $B_1 = [\mu_{B'}(x), 1 - \nu_{B'}(x)]$ and $B_2 = [\nu_{B'}(x), 1 - \mu_{B'}(x)]$ satisfying $\mu_{B'}(x) + \nu_{B'}(x) \leqslant 1$ can be written as an IFS $B' = \{< x, \mu_{B'}(x), \nu_{B'}(x) > |x \in X\}$.

The following operations on intervals of positive real numbers are due to [32,33,34]. Let $M_1 = [l_1, u_1]$, $M_2 = [l_2, u_2]$. Then $[l_1, u_1] + [l_2, u_2] = [l_1 + l_2, u_1 + u_2]$; $[l_1, u_1] - [l_2, u_2] = [l_1 - u_2, u_1 - l_2]$; $[l_1, u_1] \cdot [l_2, u_2] = [l_1 l_2, u_1 u_2]$; $[l_1, u_1]/[l_2, u_2] = [l_1/u_2, u_1/l_2]$. Any $a \in R$ can be denoted as $a = [a, a]$, and if $[l_1, u_1] \geqslant a$, then $l_1 \geqslant a, u_1 \geqslant a$.

2.2. Three kinds of preference relations

Let $N = \{1, 2, \ldots, n\}$, $M = \{1, 2, \ldots, m\}$ and $n \geqslant 3$. If a preference relation $A = (a_{ij})_{n\times n}$ satisfies $a_{ii} = 0.5, a_{ij} + a_{ji} = 1, a_{ij} > 0, i, j \in N$, then A is called a fuzzy preference relation (FPR). A FPR $A = (a_{ij})_{n\times n}$ is multiplicative consistent [35], if there exists a priority vector $V = (v_1\ v_2\ \ldots\ v_n)^T$ such that $a_{ij} = 1/(1 + v_j/v_i) = v_i/(v_i + v_j), i, j \in N$. Let $S = \{s_1, \ldots, s_n\}$ be an alternative set. If a preference relation $R' = (r'_{ij})_{n\times n}$ satisfies $r'_{ii} = [0.5, 0.5]$, $r'_{ijl} + r'_{jiu} = r'_{iju} + r'_{jil} = 1$, then R is called an interval preference relation (IPR) [36]. Here, $r'_{ij} = [r'_{ijl}, r'_{iju}]$ denotes the degree range to which the alternative s_i is preferred to the alternative s_j, $i, j \in N$. If

$r'_{ii} = [0.5, 0.5]$, then there is no difference between s_i and s_j; if $r'_{ij} > [0.5, 0.5]$, then s_i is preferred to s_j; and if $r'_{ij} < [0.5, 0.5]$, then s_j is preferred to s_i. An IPR $R' = (r'_{ij})_{n\times n}$ is multiplicative consistent [37], if there exists a priority vector $\Omega = (\omega_1 \; \ldots \; \omega_n)^T = ([\omega_{1l}, \omega_{1u}] \; \ldots \; [\omega_{nl}, \omega_{nu}])^T$ such that $r'_{ij} = 1/(1 + \omega_j/\omega_i) = [\omega_{il}/(\omega_{il} + \omega_{ju}), \omega_{iu}/(\omega_{jl} + \omega_{iu})] \; \forall \; i, j \in N$, where $\omega_i = [\omega_{il}, \omega_{iu}], i \in N$.

An intuitionistic fuzzy preference relation [38] in S is defined as

$$\mathfrak{R} = \{< (s_i, s_j), \mu_{\mathfrak{R}}(s_i, s_j), \nu_{\mathfrak{R}}(s_i, s_j) > | (s_i, s_j) \in S \times S\}$$

where $\mu_{\mathfrak{R}} : S \times S \mapsto [0,1]$, $\nu_{\mathfrak{R}} : S \times S \mapsto [0,1]$, $\mu_{\mathfrak{R}}(s_i, s_j)$ is the degree to which s_i is preferred to s_j, and $\nu_{\mathfrak{R}}(s_i, s_j)$ is the degree to which s_i is not preferred to s_j. Moreover, the inequality $0 \leqslant \mu_{\mathfrak{R}}(s_i, s_j) + \nu_{\mathfrak{R}}(s_i, s_j) \leqslant 1$ holds for every $(s_i, s_j) \in S \times S$, $i, j \in N$. The matrix format of the intuitionistic fuzzy preference relation is expressed as follows:

Let $\mathfrak{R}$ be an intuitionistic fuzzy preference relation in S. If for all $i, j \in N$, $\mu_{ij} = \mu_{\mathfrak{R}}(s_i, s_j)$, $\nu_{ij} = \nu_{\mathfrak{R}}(s_i, s_j)$ and

$$\begin{aligned} r_{ii} &= (0.5, 0.5, 0); \\ \mu_{ij} = \nu_{ji}, \nu_{ij} &= \mu_{ji}, \pi_{ij} = \pi_{ji}; \\ \mu_{ij} + \nu_{ij} + \pi_{ij} &= 1 \end{aligned} \tag{1}$$

then

$$R = (u_{ij}, \nu_{ij}, \pi_{ij}) = \begin{pmatrix} (\mu_{11}, \nu_{11}, \pi_{11}) & (\mu_{12}, \nu_{12}, \pi_{12}) & \cdots & (\mu_{1n}, \nu_{1n}, \pi_{1n}) \\ (\mu_{21}, \nu_{21}, \pi_{21}) & (\mu_{22}, \nu_{22}, \pi_{22}) & \cdots & (\mu_{2n}, \nu_{2n}, \pi_{2n}) \\ \vdots & \vdots & \cdots & \vdots \\ (\mu_{n1}, \nu_{n1}, \pi_{n1}) & (\mu_{n2}, \nu_{n2}, \pi_{n2}) & \cdots & (\mu_{nn}, \nu_{nn}, \pi_{nn}) \end{pmatrix}$$

is called an intuitionistic fuzzy judgment matrix (also called an IFPR [15,16]).

For all $i, j \in N$, μ_{ij} are the degree to which s_i being preferred to s_j, and ν_{ij} are the degree to which s_i being not preferred to s_j, and the intuitionistic indices π_{ij} are such that the larger π_{ij} the higher hesitation margin of the degree to which s_i being preferred to s_j. In the process of decision making, the DM can increase his evaluation by adding the value of the intuitionistic index [39,40]. This means that his/her judgment actually lies in the closed intervals $[\mu_{ij}, \mu_{ij} + \pi_{ij}]$ and $[\nu_{ij}, \nu_{ij} + \pi_{ij}]$.

2.3. *The relationship between the IFPR and the IPR*

The IFPR can be split into three matrices as follows:

$$u = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nn} \end{pmatrix};$$

$$v = \begin{pmatrix} \nu_{11} & \nu_{12} & \cdots & \nu_{1n} \\ \nu_{21} & \nu_{22} & \cdots & \nu_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \nu_{n1} & \nu_{n2} & \cdots & \nu_{nn} \end{pmatrix};$$

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1n} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \pi_{n1} & \pi_{n2} & \cdots & \pi_{nn} \end{pmatrix}.$$

If we combine u with π, and combine v with π, we can derive two interval matrices as follows:

$$A = (a_{ij}) = (\mu_{ij}, p_{ij}) = \begin{pmatrix} [\mu_{11}, p_{11}] & [\mu_{12}, p_{12}] & \cdots & [\mu_{1n}, p_{1n}] \\ [\mu_{21}, p_{21}] & [\mu_{22}, p_{22}] & \cdots & [\mu_{2n}, p_{2n}] \\ \vdots & \vdots & \cdots & \vdots \\ [\mu_{n1}, p_{n1}] & [\mu_{n2}, p_{n2}] & \cdots & [\mu_{nn}, p_{nn}] \end{pmatrix};$$

$$B = (b_{ij}) = (\nu_{ij}, q_{ij}) = \begin{pmatrix} [\nu_{11}, q_{11}] & [\nu_{12}, q_{12}] & \cdots & [\nu_{1n}, q_{1n}] \\ [\nu_{21}, q_{21}] & [\nu_{22}, q_{22}] & \cdots & [\nu_{2n}, q_{2n}] \\ \vdots & \vdots & \cdots & \vdots \\ [\nu_{n1}, q_{n1}] & [\nu_{n2}, q_{n2}] & \cdots & [\nu_{nn}, q_{nn}] \end{pmatrix},$$

where we denote

$$p_{ij} = 1 - \nu_{ij}, q_{ij} = 1 - \mu_{ij}. \tag{2}$$

By Eqs. (1) and (2), we have that

$$[\mu_{ii}, p_{ii}] = [0.5, 0.5], \mu_{ij} + p_{ji} = p_{ij} + \mu_{ji} = 1, i, j \in N, \tag{3}$$

$$[\nu_{ii}, q_{ii}] = [0.5, 0.5], \nu_{ij} + q_{ji} = q_{ij} + \nu_{ji} = 1, i, j \in N, \tag{4}$$

which imply that both A and B are IPRs.

A and B can be regarded as the decomposed matrices of the IFPR R. That is to say, the interval $[\mu_{ij}, p_{ij}]$ can be regarded as the range of the degree to which s_i is preferred to s_j, and $[\nu_{ij}, q_{ij}]$ the range of degree to which s_i is not preferred to s_j [39,40].

Consider again the IPRs A and B with the conditions (3) and (4) holding. Let $\pi_{ij} = p_{ij} - \mu_{ij}$. Eqs. (2), (3) and (4) actually imply that

$$\mu_{ij} = \nu_{ji}, \nu_{ij} = \mu_{ji}, i, j \in N, \quad (5)$$

$$p_{ij} = q_{ji}, q_{ij} = p_{ji}, i, j \in N, \quad (6)$$

$$\pi_{ij} = \pi_{ji}, i, j \in N, \quad (7)$$

and

$$\mu_{ij} + \nu_{ij} + \pi_{ij} = 1, i, j \in N. \quad (8)$$

In consequence, the IFPR R can be considered as a combination of the IPRs A and B.

This discussion leads to the following definition.

The IPRs $A = (a_{ij})_{n\times n} = ([\mu_{ij}, p_{ij}])_{n\times n}$ and $B = (b_{ij})_{n\times n} = ([\nu_{ij}, q_{ij}])_{n\times n}$ satisfying the conditions (2), (3) and (4) are called the equivalent matrices of the IFPR R.

3. Optimal models of the incomplete IFPR, IPR and FPR

3.1. The priority of the multiplicative consistent IFPR

Consider the equivalent matrices $A = (a_{ij})_{n\times n} = ([\mu_{ij}, p_{ij}])_{n\times n}$ and $B = (b_{ij})_{n\times n} = ([\nu_{ij}, q_{ij}])_{n\times n}$ of the IFPR $R = (r_{ij})_{n\times n} = (\mu_{ij}, \nu_{ij}, \pi_{ij})_{n\times n}$, which satisfy conditions (2), (3) and (4). Let $\Omega = (\omega_1 \ \dots \ \omega_n)^T = ([\omega_{1l}, \omega_{1u}] \ \dots \ [\omega_{nl}, \omega_{nu}])^T$ be the priority vector of the multiplicative consistent IPR A. Then

$$\begin{aligned} a_{ij} &= [\mu_{ij}, p_{ij}] = 1/(1+\omega_j/\omega_i) \\ &= [\frac{\omega_{il}}{\omega_{il}+\omega_{ju}}, \frac{\omega_{iu}}{\omega_{jl}+\omega_{iu}}], i, j \in N. \end{aligned} \quad (9)$$

That is

$$\mu_{ij} = \frac{\omega_{il}}{\omega_{il}+\omega_{ju}}, i, j \in N, \quad (10)$$

$$p_{ij} = \frac{\omega_{iu}}{\omega_{jl}+\omega_{iu}}, i, j \in N. \quad (11)$$

By Eqs. (2) and (11), we easily get

$$\nu_{ij} = 1 - p_{ij} = 1 - \frac{\omega_{iu}}{\omega_{jl}+\omega_{iu}} = \frac{\omega_{jl}}{\omega_{jl}+\omega_{iu}}, i, j \in N. \quad (12)$$

From Eqs. (10) and (11),

$$\pi_{ij} = p_{ij} - \mu_{ij} = \frac{\omega_{iu}}{\omega_{jl}+\omega_{iu}} - \frac{\omega_{il}}{\omega_{il}+\omega_{ju}}, i, j \in N. \quad (13)$$

Eqs.(12) and (13) mean that $\nu_{ij}, \pi_{ij}, i, j \in N$ can be represented by the priority vector $\Omega = (\omega_1 \ \dots \ \omega_n)^T$ of A as well.

If we let $\Omega = (\omega_1 \ \dots \ \omega_n)^T$ be the priority vector of the consistent IPR A, then the membership degree μ_{ij}, the nonmembership degree ν_{ij} and the intuitionistic fuzzy index π_{ij} of R can be derived from Eqs. (10), (12) and (13), respectively. As in Section 2.3, we can transform the interval vector Ω into intuitionistic fuzzy numbers

$$\begin{aligned} \zeta \;=\; & ((\omega_{1l}, 1-\omega_{1u}, \omega_{1u}-\omega_{1l}) \ \dots \ (\omega_{il}, 1-\omega_{iu}, \\ & \omega_{iu}-\omega_{il}) \ \dots \ (\omega_{nl}, 1-\omega_{nu}, \omega_{nu}-\omega_{nl}))^T, \end{aligned}$$

where ω_{il} is the membership degree of the importance (weight) of s_i, $1-\omega_{iu}$ the nonmembership degree of the importance (weight) of s_i, and $\omega_{iu}-\omega_{il}$ the hesitation degree of the importance (weight) of s_i, $i \in N$ [39].

An IFPR $R = (r_{ij})_{n\times n}$ is multiplicative consistent if there exists a vector $\zeta = (\zeta_1 \ \dots \ \zeta_n)^T$ such that Eqs. (10), (12) and (13) hold, where $\zeta_i = (\omega_{il}, 1-\omega_{iu}, \omega_{iu}-\omega_{il}), i \in N$. ζ is called the priority vector of the multiplicative consistent IFPR R.

Eqs. (10) and (12) are called the consistent conditions of the multiplicative consistent IFPR R because Eq. (13) is derived from Eqs. (10) and (12).

3.2. The optimal models of the incomplete IFPR

Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of alternatives and $d = \{d_1, d_2, \dots, d_m\}$ a set of DMs. The preferences of the DMs on X are described by the IFPRs as follows:

$$\widetilde{R} = (\widetilde{r}_{ijs})_{n\times n}$$
$$= \begin{pmatrix} (0.5,0.5,0) & \left\{\begin{array}{c}(\mu_{121},\nu_{121},\pi_{121})\\ \vdots \\ (\mu_{12\delta_{12}},\nu_{12\delta_{12}},\pi_{12\delta_{12}})\end{array}\right\} & \cdots & \left\{\begin{array}{c}(\mu_{1n1},\nu_{121},\pi_{1n1})\\ \vdots \\ (\mu_{1n\delta_{1n}},\nu_{1n\delta_{1n}},\pi_{1n\delta_{1n}})\end{array}\right\} \\ \left\{\begin{array}{c}(\mu_{211},\nu_{211},\pi_{211})\\ \vdots \\ (\mu_{21\delta_{21}},\nu_{21\delta_{21}},\pi_{21\delta_{21}})\end{array}\right\} & (0.5,0.5,0) & \cdots & \left\{\begin{array}{c}(\mu_{2n1},\nu_{2n1},\pi_{2n1})\\ \vdots \\ (\mu_{2n\delta_{2n}},\nu_{2n\delta_{2n}},\pi_{2n\delta_{2n}})\end{array}\right\} \\ \vdots & \vdots & \cdots & \vdots \\ \left\{\begin{array}{c}(\mu_{n11},\nu_{n11},\pi_{n11})\\ \vdots \\ (\mu_{n1\delta_{n1}},\nu_{n1\delta_{n1}},\pi_{n1\delta_{n1}})\end{array}\right\} & \left\{\begin{array}{c}(\mu_{n21},\nu_{n21},\pi_{n21})\\ \vdots \\ (\mu_{n2\delta_{n2}},\nu_{n2\delta_{n2}},\pi_{n2\delta_{n2}})\end{array}\right\} & \cdots & (0.5,0.5,0) \end{pmatrix},$$

where $\widetilde{r}_{ijs} = (\mu_{ijs},\nu_{ijs},\pi_{ijs})$ are the elements of the IFPR $\widetilde{R}$ with $\mu_{ijs} = \nu_{jis}, \nu_{ijs} = \mu_{jis}, \pi_{ijs} = \pi_{jis}, \mu_{ijs} + \nu_{ijs} + \pi_{ijs} = 1 \ \forall \ i,j \in N, i \neq j, s = 1,2,\ldots,\delta_{ij}$, and δ_{ij}, $0 \leqslant \delta_{ij} \leqslant m$ represents the number of DMs estimating the preference degree of alternative x_i over x_j. It is clear that $\delta_{ij} = \delta_{ji}$. If there exist $i_0, j_0 \in N$ such that $0 < \delta_{i_0 j_0} < m$, then $m - \delta_{i_0 j_0}$ DMs do not estimate the preference degree between alternatives x_{i_0} and x_{j_0}; if there exists $i_0, j_0 \in N$ such that $\delta_{i_0 j_0} = 0$, then no DM estimates the preference degree between alternatives x_{i_0} and x_{j_0}, and we denote $\widetilde{r}_{i_0 j_0 s} = -$. This means that the element $\widetilde{r}_{i_0 j_0 s}$ in $\widetilde{R}$ is absent and $\widetilde{R}$ are incomplete IFPRs. If for all $i,j \in N$, $\delta_{ij} = m$, then all the DMs decide on the preference between x_i and x_j, and $\widetilde{R}$ are complete IFPRs.

Suppose that all DMs hold the same degree of preference of alternative x_i over $x_j \ \forall \ i,j \in N$. For the given multiplicative consistent IFPRs $\widetilde{R} = (\mu_{ijs},\nu_{ijs},\pi_{ijs})_{n\times n}$, there must exist a priority vector $\zeta = ((\omega_{1l}, 1-\omega_{1u}, \omega_{1u}-\omega_{1l}) \ \ldots \ (\omega_{il}, 1-\omega_{iu}, \omega_{iu}-\omega_{il}) \ \ldots \ (\omega_{nl}, 1-\omega_{nu}, \omega_{nu}-\omega_{nl}))^T$ such that

$$\mu_{ijs} = \frac{\omega_{il}}{\omega_{il}+\omega_{ju}}, \tag{14}$$

$$\nu_{ijs} = 1 - \frac{\omega_{iu}}{\omega_{jl}+\omega_{iu}} = \frac{\omega_{jl}}{\omega_{jl}+\omega_{iu}}, \tag{15}$$

where $0 < \omega_{il} \leqslant \omega_{iu} < 1, i,j \in N$.

Eqs. (14) and (15) are equivalent to the following equations:

$$\mu_{ijs}\omega_{ju} - (1-\mu_{ijs})\omega_{il} = 0, \tag{16}$$

$$\nu_{ijs}\omega_{iu} - (1-\nu_{ijs})\omega_{jl} = 0. \tag{17}$$

Eqs. (16) and (17) are actually the ideal cases. In reality, it is hard for a DM be consistent, and different DMs may present different judgments. In consequence, Eqs. (16) and (17) may not hold. Consider the following deviation functions:

$$\varepsilon_{ijs} = [\mu_{ijs}\omega_{ju} - (1-\mu_{ijs})\omega_{il}]^2, \tag{18}$$

$$\gamma_{ijs} = [\nu_{ijs}\omega_{iu} - (1-\nu_{ijs})\omega_{jl}]^2. \tag{19}$$

It is clear that small deviation functions represent better consistency of judgment. In order to get the optimal priority vector of the inconsistent IFPRs, we introduce a least squares optimal model as follows:

$$\begin{aligned} min\ J = \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{s=1}^{\delta_{ij}} & [\mu_{ijs}\omega_{ju} - (1-\mu_{ijs})\omega_{il}]^2 \\ & +[\nu_{ijs}\omega_{iu} - (1-\nu_{ijs})\omega_{jl}]^2 \end{aligned}$$
$$s.t. \quad \begin{cases} \omega_{il} + \sum\limits_{j=1,j\neq i}^{n} \omega_{ju} \geqslant 1, & i \in N; \\ \omega_{iu} + \sum\limits_{j=1,j\neq i}^{n} \omega_{jl} \leqslant 1, & i \in N; \\ \omega_{iu} - \omega_{il} \geqslant 0, & i \in N; \\ \omega_{iu} \geqslant 0, \omega_{il} \geqslant 0, & i \in N. \end{cases} \tag{20}$$

The two constraints $\omega_{il} + \sum_{j=1, j\neq i}^{n} \omega_{ju} \geqslant 1$ and $\omega_{iu} + \sum_{j=1, j\neq i}^{n} \omega_{jl} \leqslant 1$ are the normalization constraints on the interval vector Ω [41].

In model (20), if $\delta_{ij} = m \ \forall \ i,j \in N$, then we get a collective priority model of the IFPRs presented by m DMs with complete information. If $\delta_{ij} = 1 \forall \ i,j \in N$, then we get a priority model of individual IFPR with complete information. If there exist $i_0, j_0 \in N$ such that $0 < \delta_{i_0 j_0} < m$, then we get a collective priority model of the IFPRs presented by m DMs with incomplete information.

3.3. The relation between the priority of the IFPRs and that of the IPRs

Suppose that the preferences on X are described by the following IPR:

$$\breve{A} = \begin{pmatrix} [0.5, 0.5] & \left\{ \begin{matrix} [\widetilde{\mu}_{121}, \widetilde{p}_{121}] \\ \vdots \\ [\widetilde{\mu}_{12\delta_{12}}, \widetilde{p}_{12\delta_{12}}] \end{matrix} \right\} & \cdots & \left\{ \begin{matrix} [\widetilde{\mu}_{1n1}, \widetilde{p}_{1n1}] \\ \vdots \\ [\widetilde{\mu}_{1n\delta_{1n}}, \widetilde{p}_{1n\delta_{1n}}] \end{matrix} \right\} \\ \left\{ \begin{matrix} [\widetilde{\mu}_{211}, \widetilde{p}_{211}] \\ \vdots \\ [\widetilde{\mu}_{21\delta_{21}}, \widetilde{p}_{21\delta_{21}}] \end{matrix} \right\} & [0.5, 0.5] & \cdots & \left\{ \begin{matrix} [\widetilde{\mu}_{2n1}, \widetilde{p}_{2n1}] \\ \vdots \\ [\widetilde{\mu}_{2n\delta_{2n}}, \widetilde{p}_{2n\delta_{2n}}] \end{matrix} \right\} \\ \vdots & \vdots & \cdots & \vdots \\ \left\{ \begin{matrix} [\widetilde{\mu}_{n11}, \widetilde{p}_{n11}] \\ \vdots \\ [\widetilde{\mu}_{n1\delta_{n1}}, \widetilde{p}_{n1\delta_{n1}}] \end{matrix} \right\} & \left\{ \begin{matrix} [\widetilde{\mu}_{n1}, \widetilde{p}_{n21}] \\ \vdots \\ [\widetilde{\mu}_{n2\delta_{n2}}, \widetilde{p}_{n2\delta_{n2}}] \end{matrix} \right\} & \cdots & [0.5, 0.5] \end{pmatrix}$$

where $[\widetilde{\mu}_{ijs}, \widetilde{p}_{ijs}]$ are the elements of the IPR $\breve{A}$ with $\widetilde{\mu}_{ijs} + \widetilde{p}_{jis} = 1, \widetilde{\mu}_{jis} + \widetilde{p}_{ijs} = 1 \forall \ i,j \in N, i \neq j, s = 1,2\ldots,\delta_{ij}$, and δ_{ij}, $0 \leqslant \delta_{ij} \leqslant m$ represents the number of DMs estimating the preference degree of alternative x_i over x_j.

Let $\widetilde{\zeta} = ([\widetilde{\omega}_{1l}, \widetilde{\omega}_{1u}] \ldots [\widetilde{\omega}_{il}, \widetilde{\omega}_{iu}] \ \ldots \ [\widetilde{\omega}_{nl}, \widetilde{\omega}_{nu}])^T$ be the priority vector of the multiplicative consistent IPR $\breve{A}$. Then

$$\widetilde{\mu}_{ijs} = \frac{\widetilde{\omega}_{il}}{\widetilde{\omega}_{il} + \widetilde{\omega}_{ju}}, \tag{21}$$

$$\widetilde{p}_{ijs} = \frac{\widetilde{\omega}_{iu}}{\widetilde{\omega}_{jl} + \widetilde{\omega}_{iu}}, \tag{22}$$

where $0 < \widetilde{\omega}_{il} \leqslant \widetilde{\omega}_{iu} \leqslant 1, i,j \in N$. Eqs. (21) and (22) are equivalent to the following equations.

$$\widetilde{\mu}_{ijs}\widetilde{\omega}_{ju} - (1 - \widetilde{\mu}_{ijs})\widetilde{\omega}_{il} = 0, \tag{23}$$

$$\widetilde{p}_{ijs}\widetilde{\omega}_{jl} - (1 - \widetilde{p}_{ijs})\widetilde{\omega}_{iu} = 0. \tag{24}$$

Let

$$\widetilde{\varepsilon}_{ijs} = [\widetilde{\mu}_{ijs}\widetilde{\omega}_{ju} - (1 - \widetilde{\mu}_{ijs})\widetilde{\omega}_{il}]^2, \tag{25}$$

$$\widetilde{\gamma}_{ijs} = [\widetilde{p}_{ijs}\widetilde{\omega}_{jl} - (1 - \widetilde{p}_{ijs})\widetilde{\omega}_{iu}]^2. \tag{26}$$

As in Section 3.2, we introduce a least squares optimal model to get the priority of the inconsistent IPRs:

$$\begin{aligned} min\ J = & \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \sum_{s=1}^{\delta_{ij}} [\widetilde{\mu}_{ijs}\widetilde{\omega}_{ju} - (1 - \widetilde{\mu}_{ijs})\widetilde{\omega}_{il}]^2 \\ & + [\widetilde{p}_{ijs}\widetilde{\omega}_{jl} - (1 - \widetilde{p}_{ijs})\widetilde{\omega}_{iu}]^2 \\ s.t. \quad & \begin{cases} \widetilde{\omega}_{il} + \sum_{j=1, j\neq i}^{n} \widetilde{\omega}_{ju} \geqslant 1, & i \in N; \\ \widetilde{\omega}_{iu} + \sum_{j=1, j\neq i}^{n} \widetilde{\omega}_{jl} \leqslant 1, & i \in N; \\ \widetilde{\omega}_{iu} - \widetilde{\omega}_{il} \geqslant 0, & i \in N; \\ \widetilde{\omega}_{iu} \geqslant 0, \widetilde{\omega}_{il} \geqslant 0, & i \in N. \end{cases} \end{aligned} \tag{27}$$

Obviously, for all $i,j \in N$, if $\delta_{ij} = m$, model (27) can be regarded as a collective priority model of IPR with complete information. If $\delta_{ij} = 1$, model (27) can be regarded as an individual priority model of IPR with complete information. If there exist $i_0, j_0 \in N$ such that $0 < \delta_{i_0 j_0} < m$, then we get a collective priority model of incomplete IPR.

Given an IFPR $\widetilde{R}$ with equivalent matrices $\breve{A}$, we have $\mu_{ijs} = \widetilde{\mu}_{ijs}$, $\nu_{ijs} = 1 - \widetilde{p}_{ijs} \forall i, j \in N$. Obviously, models (20) and (27) have the same objective functions and the same constrained conditions. Thus we easily conclude the following:

Theorem 1 Given an IFPR $\widetilde{R}$ with equivalent matrices $\breve{A}$, model (20) and model (27) have the same optimal solutions.

Theorem 1 indicates that regardless of whether $\widetilde{R}$ or its equivalent matrices $\breve{A}$ are multiplicative consistent, models (20) and (27) both have the same optimal solutions.

Theorem 1 gives the relation between the optimal priority models of IFPRs and IPRs. Although models (20) and (27) seem very similar, their meanings are different. The objective function of model (20) takes into account not only the membership degree, but also the nonmembership degree, while the objective function of model (27) accounts only for the membership degree range.

3.4. *The relation between the priority of the IPRs and that of the FPRs*

For all $i, j \in N$, values $\widetilde{p}_{ijs}$ in $\breve{A} = ([\widetilde{\mu}_{ijs}, \widetilde{p}_{ijs}])_{n\times n}$ denote the maximum degree to which alternative x_i is preferred to x_j. Consider the FPR $\overline{A} = (\overline{a}_{ijs})_{n\times n}$, where $\begin{cases} \overline{a}_{ijs} = \widetilde{p}_{ijs},\ if\ i<j; \\ \overline{a}_{ijs} = \widetilde{\mu}_{ijs},\ if\ i>j; \\ \overline{a}_{iis} = 0.5,\ \ if\ i=j. \end{cases}$ We call $\overline{A}$ the maximum degree preference relation of $\breve{A}$, and $\overline{a}_{ijs}$ maximum membership judgment preference.

Let $\overline{v} = (\overline{v}_1 \dots\ \overline{v}_i\ \dots\ \overline{v}_n)^T$ be the priority vector of the multiplicative consistent FPRs $\overline{A}$. Then

$$\overline{a}_{ijs} = \frac{\overline{v}_i}{\overline{v}_i + \overline{v}_j}. \tag{28}$$

Eq. (28) is equivalent to the following:

$$\overline{a}_{ijs}\overline{v}_j - \overline{a}_{jis}\overline{v}_i = 0. \tag{29}$$

Let

$$\overline{\varepsilon}_{ijs} = (\overline{a}_{ijs}\overline{v}_j - \overline{a}_{jis}\overline{v}_i)^2. \tag{30}$$

As in Sections 3.2 and 3.3, we introduce the least squares optimal model of FPRs $\overline{A}$ [35]:

$$\begin{aligned} &min\ J = \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{s=1}^{\delta_{ij}}(\overline{a}_{ijs}\overline{v}_j - \overline{a}_{jis}\overline{v}_i)^2 \\ &s.t. \quad \begin{cases} \sum_{i=1}^{n}\overline{v}_i = 1, \\ \overline{v}_i > 0, i \in N. \end{cases} \end{aligned} \tag{31}$$

Obviously, for all $i, j \in N$, if $\delta_{ij} = m$, then model (31) can be regarded as a collective priority model of FPRs with complete information; if $\delta_{ij} = 1$, then model (31) can be regarded as an individual priority model of FPR with complete information; and if there exist $i_0, j_0 \in N$ such that $0 < \delta_{i_0 j_0} < m$, then we get a collective priority model of FPRs with incomplete information. The optimal solution of model (31) is:

$$\overline{V} = (\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_n)^T = Q^{-1}e/e^TQ^{-1}e, \tag{32}$$

where

$$Q = \begin{pmatrix} \sum_{i=1,i\neq 1}^{n}\sum_{s=1}^{\delta_{1i}}\overline{a}_{i1s}^2 & -\sum_{s=1}^{\delta_{12}}\overline{a}_{12s}\overline{a}_{21s} & \cdots & -\sum_{s=1}^{\delta_{1n}}\overline{a}_{1ns}\overline{a}_{n1s} \\ -\sum_{s=1}^{\delta_{21}}\overline{a}_{12s}\overline{a}_{21s} & \sum_{i=1,i\neq 2}^{n}\sum_{s=1}^{\delta_{2i}}\overline{a}_{i2s}^2 & \cdots & -\sum_{s=1}^{\delta_{2n}}\overline{a}_{2ns}\overline{a}_{n2s} \\ \cdots & \cdots & \cdots & \cdots \\ -\sum_{s=1}^{\delta_{n1}}\overline{a}_{1ns}\overline{a}_{n1s} & -\sum_{s=1}^{\delta_{n2}}\overline{a}_{n2s}\overline{a}_{2ns} & \cdots & \sum_{i=1,i\neq n}^{n}\sum_{s=1}^{\delta_{ni}}\overline{a}_{ins}^2 \end{pmatrix};$$

$$e = \begin{pmatrix} 1\ 1\ \dots\ 1 \end{pmatrix}^T.$$

4. Numerical examples

Example 1 Suppose that there are three DMs providing the following incomplete IFPRs $\{\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3\}$ on a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$.

$$\widetilde{R}_1 = \begin{pmatrix} (0.5,0.5,0) & (0.1,0.6,0.3) & - & - \\ (0.6,0.1,0.3) & (0.5,0.5,0) & (0.8,0.2,0) & - \\ - & (0.2,0.8,0) & (0.5,0.5,0) & (0.8,0.1,0.1) \\ - & - & (0.1,0.8,0.1) & (0.5,0.5,0) \end{pmatrix};$$

$$\widetilde{R}_2 = \begin{pmatrix} (0.5,0.5,0) & (0.2,0.7,0.1) & - & - \\ (0.7,0.2,0.1) & (0.5,0.5,0) & (0.7,0.1,0.2) & - \\ - & (0.1,0.7,0.2) & (0.5,0.5,0) & - \\ - & - & - & (0.5,0.5,0) \end{pmatrix};$$

$$\widetilde{R}_3 = \begin{pmatrix} (0.5,0.5,0) & (0.1,0.6,0.3) & (0.7,0.1,0.2) & - \\ (0.6,0.1,0.3) & (0.5,0.5,0) & - & (0.8,0.1,0.1) \\ (0.1,0.7,0.2) & - & (0.5,0.5,0) & - \\ - & (0.1,0.8,0,1) & - & (0.5,0.5,0) \end{pmatrix}.$$

Step 1: Using model (20), we first construct the optimal model as follows:

$$
\begin{aligned}
\min J_1 = {} & (0.1\omega_{2u}-0.9\omega_{1l})^2+(0.6\omega_{1u}-0.4\omega_{2l})^2\\
&+(0.2\omega_{2u}-0.8\omega_{1l})^2+(0.7\omega_{1u}-0.3\omega_{2l})^2\\
&+(0.1\omega_{2u}-0.9\omega_{1l})^2+(0.6\omega_{1u}-0.4\omega_{2l})^2\\
&+(0.7\omega_{3u}-0.3\omega_{1l})^2+(0.1\omega_{1u}-0.9\omega_{3l})^2\\
&+(0.6\omega_{1u}-0.4\omega_{2l})^2+(0.1\omega_{2u}-0.9\omega_{1l})^2\\
&+(0.7\omega_{1u}-0.3\omega_{2l})^2+(0.2\omega_{2u}-0.8\omega_{1l})^2\\
&+(0.6\omega_{1u}-0.4\omega_{2l})^2+(0.1\omega_{2u}-0.9\omega_{1l})^2\\
&+(0.8\omega_{3u}-0.2\omega_{2l})^2+(0.2\omega_{2u}-0.8\omega_{3l})^2\\
&+(0.7\omega_{3u}-0.3\omega_{2l})^2+(0.1\omega_{2u}-0.9\omega_{3l})^2\\
&+(0.8\omega_{4u}-0.2\omega_{2l})^2+(0.1\omega_{2u}-0.9\omega_{4l})^2\\
&+(0.1\omega_{1u}-0.9\omega_{3l})^2+(0.7\omega_{3u}-0.3\omega_{1l})^2\\
&+(0.2\omega_{2u}-0.8\omega_{3l})^2+(0.8\omega_{3u}-0.2\omega_{2l})^2\\
&+(0.1\omega_{2u}-0.9\omega_{3l})^2+(0.7\omega_{3u}-0.3\omega_{2l})^2\\
&+(0.8\omega_{4u}-0.2\omega_{3l}))^2+(0.1\omega_{3u}-0.9\omega_{4l})^2\\
&+(0.1\omega_{2u}-0.9\omega_{4l})^2+(0.8\omega_{4u}-0.2\omega_{2l})^2\\
&+(0.1\omega_{3u}-0.9\omega_{4l})^2+(0.8\omega_{4u}-0.2\omega_{3l})^2
\end{aligned}
$$

$$
s.t.\quad \begin{cases}
\omega_{1l}+\omega_{2u}+\omega_{3u}+\omega_{4u}\geqslant 1,\\
\omega_{2l}+\omega_{1u}+\omega_{3u}+\omega_{4u}\geqslant 1,\\
\omega_{3l}+\omega_{1u}+\omega_{2u}+\omega_{4u}\geqslant 1,\\
\omega_{4l}+\omega_{1u}+\omega_{2u}+\omega_{3u}\geqslant 1,\\
\omega_{1u}+\omega_{2l}+\omega_{3l}+\omega_{4l}\leqslant 1,\\
\omega_{2u}+\omega_{1l}+\omega_{3l}+\omega_{4l}\leqslant 1,\\
\omega_{3u}+\omega_{1l}+\omega_{2l}+\omega_{4l}\leqslant 1,\\
\omega_{4u}+\omega_{1l}+\omega_{2l}+\omega_{3l}\leqslant 1,\\
\omega_{1u}-\omega_{1l}\geqslant 0,\\
\omega_{2u}-\omega_{2l}\geqslant 0,\\
\omega_{3u}-\omega_{3l}\geqslant 0,\\
\omega_{4u}-\omega_{4l}\geqslant 0,\\
\omega_{il}\geqslant 0,\omega_{iu}\geqslant 0, i,j=1,2,3,4.
\end{cases}
\tag{33}
$$

Step 2: We use the $'Matlab\ Optimization\ Toolbox'$ to obtain the solutions to (33):

$\omega_{1l}=0.1132;\omega_{1u}=0.2876;\omega_{2l}=0.4791;\omega_{2u}=0.6536;$

$\omega_{3l}=0.0886;\omega_{3u}=0.1418;\omega_{4l}=0.0442;\omega_{4u}=0.0914.$

Step 3: The priority vector of $\widetilde{R}$ is found to be $((0.1132,0.7124,0.1744)\ (0.4791,0.3464,0.1745)\ (0.0886,0.8582,0.0532)\ (0.0442,0.9086,0.0472))^T$.

Step 4: Using the comparative method of two intuitionistic fuzzy numbers [8,9], the optimal ranking order of the alternatives is found to be $x_2 \succ x_1 \succ x_3 \succ x_4$.

Example 2 Consider the equivalent matrices $\breve{A}_i$ of $\widetilde{R}_i$, $i=1,2,3$.

$$
\breve{A}_1=\begin{pmatrix}
[0.5,0.5] & [0.1,0.4] & - & -\\
[0.6,0.9]) & [0.5,0.5] & [0.8,0.8] & -\\
- & [0.2,0.2] & [0.5,0.5] & [0.8,0.9]\\
- & - & [0.1,0.2] & [0.5,0.5]
\end{pmatrix};
$$

$$
\breve{A}_2=\begin{pmatrix}
[0.5,0.5] & [0.2,0.3] & - & -\\
[0.7,0.8] & [0.5,0.5] & [0.7,0.9] & -\\
- & [0.1,0.3] & [0.5,0.5] & -\\
- & - & - & [0.5,0.5]
\end{pmatrix};
$$

$$
\breve{A}_3=\begin{pmatrix}
[0.5,0.5] & [0.1,0.4]) & [0.7,0.9] & -\\
[0.6,0.9] & [0.5,0.5] & - & [0.8,0.9]\\
[0.1,0.3] & - & [0.5,0.5] & -\\
- & [0.1,0.2] & - & [0.5,0.5]
\end{pmatrix}.
$$

From (27) we construct the following optimal model:

$$
\begin{aligned}
\min J_2 = {} & (0.1\omega_{2u}-0.9\omega_{1l})^2+(0.4\omega_{2l}-0.6\omega_{1u})^2\\
&+(0.2\omega_{2u}-0.8\omega_{1l})^2+(0.3\omega_{2l}-0.7\omega_{1u})^2\\
&+(0.1\omega_{2u}-0.9\omega_{1l})^2+(0.4\omega_{2l}-0.6\omega_{1u})^2\\
&+(0.7\omega_{3u}-0.3\omega_{1l})^2+(0.9\omega_{3l}-0.1\omega_{1u})^2\\
&+(0.6\omega_{1u}-0.4\omega_{2l})^2+(0.9\omega_{1l}-0.1\omega_{2u})^2\\
&+(0.7\omega_{1u}-0.3\omega_{2l})^2+(0.8\omega_{1l}-0.2\omega_{2u})^2\\
&+(0.6\omega_{1u}-0.4\omega_{2l})^2+(0.9\omega_{1l}-0.1\omega_{2u})^2\\
&+(0.8\omega_{3u}-0.2\omega_{2l})^2+(0.8\omega_{3l}-0.2\omega_{2u})^2\\
&+(0.7\omega_{3u}-0.3\omega_{2l})^2+(0.9\omega_{3l}-0.1\omega_{2u})^2\\
&+(0.8\omega_{4u}-0.2\omega_{2l})^2+(0.9\omega_{4l}-0.1\omega_{2u})^2\\
&+(0.1\omega_{1u}-0.9\omega_{3l})^2+(0.3\omega_{1l}-0.7\omega_{3u})^2\\
&+(0.2\omega_{2u}-0.8\omega_{3l})^2+(0.2\omega_{2l}-0.8\omega_{3u})^2\\
&+(0.1\omega_{2u}-0.9\omega_{3l})^2+(0.3\omega_{2l}-0.7\omega_{3u})^2\\
&+(0.8\omega_{4u}-0.2\omega_{3l})^2+(0.9\omega_{4l}-0.1\omega_{3u})^2\\
&+(0.1\omega_{2u}-0.9\omega_{4l})^2+(0.2\omega_{2l}-0.8\omega_{4u})^2\\
&+(0.1\omega_{3u}-0.9\omega_{4l})^2+(0.2\omega_{3l}-0.8\omega_{4u})^2
\end{aligned}
$$

$$
s.t.\quad \begin{cases}
\omega_{1l}+\omega_{2u}+\omega_{3u}+\omega_{4u}\geqslant 1,\\
\omega_{2l}+\omega_{1u}+\omega_{3u}+\omega_{4u}\geqslant 1,\\
\omega_{3l}+\omega_{1u}+\omega_{2u}+\omega_{4u}\geqslant 1,\\
\omega_{4l}+\omega_{1u}+\omega_{2u}+\omega_{3u}\geqslant 1,\\
\omega_{1u}+\omega_{2l}+\omega_{3l}+\omega_{4l}\leqslant 1,\\
\omega_{2u}+\omega_{1l}+\omega_{3l}+\omega_{4l}\leqslant 1,\\
\omega_{3u}+\omega_{1l}+\omega_{2l}+\omega_{4l}\leqslant 1,\\
\omega_{4u}+\omega_{1l}+\omega_{2l}+\omega_{3l}\leqslant 1,\\
\omega_{1u}-\omega_{1l}\geqslant 0,\\
\omega_{2u}-\omega_{2l}\geqslant 0,\\
\omega_{3u}-\omega_{3l}\geqslant 0,\\
\omega_{4u}-\omega_{4l}\geqslant 0,\\
\omega_{il}\geqslant 0,\omega_{iu}\geqslant 0, i,j=1,2,3,4.
\end{cases}
\tag{34}
$$

Obviously this model is equivalent to model (33). Thus the solutions to (34) are:

$\omega_{1l}=0.1132;\omega_{1u}=0.2876;\omega_{2l}=0.4791;\omega_{2u}=0.6536;$

$\omega_{3l}=0.0886;\omega_{3u}=0.1418;\omega_{4l}=0.0442;\omega_{4u}=0.0914.$

The priority vector of $\breve{A}$ is found to be

$$([0.1132,0.2876]\ [0.4791,0.6536]\ [0.0886,0.1418]\ [0.0442,0.0914])^T.$$

Again, using the comparative method of two interval fuzzy numbers [36], we find

$$([0.4791, 0.6536] \underset{100\%}{\geqslant} [0.1132, 0.2876] \underset{87.43\%}{\geqslant} [0.0886, 0.1418] \underset{97.21\%}{\geqslant} [0.0442, 0.0914])^T.$$

The optimal ranking order of the alternatives is given by

$$x_2 \underset{100\%}{\succ} x_1 \underset{87.43\%}{\succ} x_3 \underset{97.21\%}{\succ} x_4.$$

Example 3 Consider the maximum degree preference relation $\overline{A}_i$ of $\breve{A}_i$, $i = 1,2,3$.

$$\overline{A}_1 = \begin{pmatrix} 0.5 & 0.4 & - & - \\ 0.6 & 0.5 & 0.8 & - \\ - & 0.2 & 0.5 & 0.9 \\ - & - & 0.1 & 0.5 \end{pmatrix};$$

$$\overline{A}_2 = \begin{pmatrix} 0.5 & 0.3 & - & - \\ 0.7 & 0.5 & 0.9 & - \\ - & 0.1 & 0.5 & - \\ - & - & - & 0.5 \end{pmatrix};$$

$$\overline{A}_3 = \begin{pmatrix} 0.5 & 0.4 & 0.9 & - \\ 0.6 & 0.5 & - & 0.9 \\ 0.1 & - & 0.5 & - \\ - & 0.1 & - & 0.5 \end{pmatrix}.$$

By model (31), the optimal model is constructed as follows:

$$\begin{aligned} min\ J_3 = & (0.6\overline{v}_1 - 0.4\overline{v}_2)^2 + (0.7\overline{v}_1 - 0.3\overline{v}_2)^2 \\ & +(0.6\overline{v}_1 - 0.4\overline{v}_2)^2 + (0.1\overline{v}_1 - 0.9\overline{v}_3)^2 \\ & +(0.4\overline{v}_2 - 0.6\overline{v}_1)^2 + (0.3\overline{v}_2 - 0.7\overline{v}_1)^2 \\ & +(0.4\overline{v}_2 - 0.6\overline{v}_1)^2 + (0.2\overline{v}_2 - 0.8\overline{v}_3)^2 \\ & +(0.1\overline{v}_2 - 0.9\overline{v}_3)^2 + (0.1\overline{v}_2 - 0.9\overline{v}_4)^2 \\ & +(0.9\overline{v}_3 - 0.1\overline{v}_1)^2 + (0.8\overline{v}_3 - 0.2\overline{v}_2)^2 \\ & +(0.9\overline{v}_3 - 0.1\overline{v}_2)^2 + (0.1\overline{v}_3 - 0.9\overline{v}_4)^2 \\ & +(0.9\overline{v}_4 - 0.1\overline{v}_2)^2 + (0.9\overline{v}_4 - 0.1\overline{v}_3)^2 \\ s.t. \quad & \begin{cases} \overline{v}_1 + \overline{v}_2 + \overline{v}_3 + \overline{v}_4 = 1, \\ \overline{v}_i > 0, i = 1,2,3,4. \end{cases} \end{aligned} \tag{35}$$

The optimal solution to model (35) is

$$\overline{V} = Q^{-1}e/e^T Q^{-1}e = (0.3266, 0.5520, 0.0799, 0.0415)^T$$

where

$$Q = \begin{pmatrix} 1.22 & -0.69 & -0.09 & 0 \\ -0.69 & 0.47 & -0.25 & -0.09 \\ -0.09 & -0.25 & 2.27 & -0.09 \\ 0 & -0.25 & -0.09 & 1.62 \end{pmatrix}$$

$$e = \begin{pmatrix} 1\ 1\ 1\ 1 \end{pmatrix}^T.$$

Also, the optimal ranking order of the alternatives is $x_2 \succ x_1 \succ x_3 \succ x_4$.

5. Conclusions

We have derived priority methods of incomplete IFPRs and IPRs based on multiplicative consistent conditions. Our theorem shows that optimal models of IFPRs and IPRs have the same solutions. The priority approaches of these two kinds of incomplete preference relations originate from incomplete FPRs, while the priority approach of incomplete FPRs comes from complete FPRs. Consequently, our results are useful not only in treating imprecise or unreliable decision making problems, but also in describing their theoretical significance:

On one hand, the optimal priority model of the incomplete IFPRs applies to the complete IFPRs. In this sense, the optimal model of the incomplete IFPRs generalizes that of the collective complete IFPRs. On the other hand, in the optimal priority model of the IFPRs, if we replace the deviation function of nonmembership judgment preference with the deviation function of maximum membership judgment preference, then we get the optimal priority model of the IPRs. Similarly, if the elements in IPRs are replaced with crisp numbers, then we get the optimal priority model of the FPRs.

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