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Stability Analysis of a Type of T-S Fuzzy Control Systems Using Off-Axis Circle Criterion

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Abstract

In this paper, based on the off-axis circle criterion, a sufficient condition with a simple graphical explanation is derived to analyze the global asymptotic stability of a type of Takagi-Sugeno (T-S) fuzzy control systems in case of different constant reference inputs. Three numerical examples are given to demonstrate how to use the proposed method in analyzing the T-S fuzzy control systems.

Keywords: T-S fuzzy control systems, stability analysis, circle criterion, off-axis circle criterion.

1. Introduction

In the history of fuzzy control theory, the Takagi-Sugeno (T-S) fuzzy model [1] is a famous landmark. For a given T-S fuzzy model, Sugeno and Kang propose an interesting fuzzy controller design approach named Parallel Distributed Compensation [2]. The corresponding stability analysis is given in Ref. 3. Furthermore, more relaxed sufficient stability conditions have been obtained to reduce their conservatism. For example, a relaxed condition is derived based on piecewise quadratic Lyapunov functions in Ref. 4. Feng presents a comprehensive survey on the recent progresses in the stability analysis issues of the T-S fuzzy systems [5]. Based on these stability and robust stability analysis, numerous systematic design schemes have been proposed to synthesize the T-S fuzzy controllers in order to guarantee the stability or performances of the

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overall T-S fuzzy control systems even in the presence of parameter uncertainties [6, 7, 8]. Most of the above work is discussed only in the time domain, and the results are mainly presented in the form of Linear Matrix Inequalities (LMIs). Some numerical optimization methods, such as interior point method, are utilized to solve the LMIs. With the popular MATLAB LMI tool box, the system analysis and controller design can be made in an efficient way. However, to some extent, the LMI-based techniques heavily depend on numerical solutions. Thus, it is indeed of great significance to find alternative techniques to analyze the T-S fuzzy control systems, which should be more straightforward as well as less numerically dependent.

As we know that the frequency response methods have been well developed and widely used by control engineers during the past decades. For example, the effect of noise in a control system can be evaluated by its frequency response. One of the most prominent features of the frequency domain-based methods is that some useful graphs, e.g., Bode and Nyquist plots, can be employed to analyze the performances of the closed-loop systems or synthesis the controllers for certain plants. Since the T-S fuzzy model usually consists of a family of local linear dynamic systems, it is valuable to graphically analyze the T-S fuzzy systems in the frequency domain. As a matter of fact, the work of exploring the fuzzy control systems in the frequency domain can be dated to Ref. 9, in which the describing function method is used to analyze the Mamdani type fuzzy control systems. More relevant references can also be found [10]. The circle criterion is applied for the stability analysis of the Mamdani type fuzzy control systems [11, 12]. For the recent contributions, interested readers can refer to Refs. 13, 14, 15, and references therein, in which different kinds of Mamdani type fuzzy control systems are considered.

A lot of similar work has also been done for the T-S fuzzy control systems. For example, the describing function method is applied in stability analysis [16]. The circle criterion is employed to analyze the stability of T-S fuzzy control systems in [17]. Unfortunately, there is no corresponding graphical interpretation in the frequency domain yet. Although the circle criterion with its graphical interpretation is used to deal with the issue of robust stability for the T-S fuzzy controller [18, 19], however, this fuzzy controller is only one special kind of nonlinear controller, and the available results cannot be directly generalized to analyze the general fuzzy control systems. In our previous work, the circle criterion is deployed to derive sufficient stability conditions for the T-S fuzzy control systems with a graphical interpretation in the senses of Lyapunov stability [20, 21]. The off-axis circle criterion [22] is also employed to analyze the same type of T-S fuzzy control systems [23]. Nevertheless, the reference inputs of all the aforementioned fuzzy control systems are always assumed to be zero. It is well known that the reference inputs, in fact, have a significant impact on the stability of the T-S fuzzy control systems. The aim of this paper is to generalize the off-axis circle criterion-based stability condition to the T-S fuzzy control systems in case of different constant reference inputs. The advantage of our method over the existing circle criterion criterion-based approach [11, 20] is that it can still guarantee the asymptotic stability under the above circumstances.

The rest of this paper is organized as follows. Section 2 defines the T-S fuzzy control system to be studied. In Section 3, a typical case study is illustrated to show the influences of the reference inputs on the stability of the fuzzy control systems. Section 4 presents the main results of the paper: a new stability condition with graphical interpretation is proposed for the T-S fuzzy control systems by using the off-axis circle criterion. Our method can guarantee the global asymptotic stability of the T-S fuzzy control systems in case of different constant reference inputs. Three simulation examples are provided to demonstrate how to use the proposed approach for the stability analysis in Section 5. Some remarks and conclusions are finally drawn in Section 6.

2. The T-S Fuzzy Control System

The structure of the T-S fuzzy control system studied in this paper is shown in Fig. 1, where FLC and $G(s)$ are the T-S fuzzy controller and a linear Single Input Single Output (SISO) plant, respectively, r is the reference input, and u and y are the outputs of the FLC and plant, respectively.

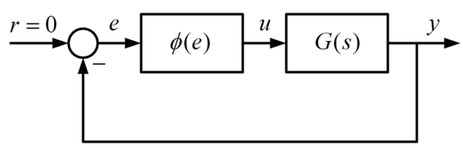

Fig. 1. The T-S fuzzy control system.

This T-S fuzzy controller consists of   reasoning rules, and the *i*-th rule is of the following form:

Rule i : If e is M_i, then $u_i = k_i e$, $i = 1, \cdots, n$, (1)

where e is the input of the fuzzy controller, u_i is the output of the local consequent function, which is actually a proportional controller, and M_i is the fuzzy set, whose corresponding membership function is denoted by μ_i. The membership functions are illustrated in Fig. 2, where $a, b > 0$.

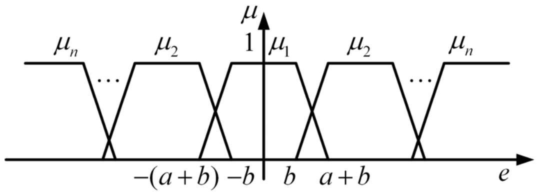

Fig. 2. Membership functions of M_i.

If $\phi(e)$ represents the nonlinear function achieved by the T-S fuzzy controller shown in Fig. 1, for the simplification of our presentation, we conclude that $\phi(e)$ belongs to the class $M_{[k_{\min}, k_{\max}]}$, if

$$k_{\min} \le \frac{\phi(e_2) - \phi(e_1)}{e_2 - e_1} \le k_{\max}, \ e_1 \ne e_2. \tag{2}$$

Lemma 1: If $\phi(e)$ represents the nonlinear function achieved by the T-S fuzzy controller in Fig. 1, and let

$$\begin{aligned} A_i = \{k_i, &\frac{(i-1)a + (2i-1)b}{a}(k_{i+1} - k_i) + k_i, \\ &\frac{(i-1)a + (2i-1)b}{a}(k_{i+1} - k_i) + (2k_{i+1} - k_i)\}, i < n, \end{aligned} \tag{3}$$

$$A_n = \{k_n\}, \tag{4}$$

$$A = \bigcup_{i=1}^{n} A_i, \tag{5}$$

and denote

$$k_{\min} = \min\{z, z \in A\}, \tag{6}$$

$$k_{\max} = \max\{z, z \in A\}. \tag{7}$$

Then, we have $\phi(e) \in M_{[k_{\min}, k_{\max}]}$.

Proof. With the fuzzy logic rules (1) and membership functions in Fig. 2, it is obvious that the mathematical expression of $\phi(e)$ is continuous, and the problem of finding the maximum and minimum of $\frac{\phi(e_2) - \phi(e_1)}{e_2 - e_1}$ for all $e_1 \ne e_2$ is equivalent to finding the maximum and mini¬mum of the derivative of $\phi(e)$. As $\phi(e)$ is odd, we only focus on $e \ge 0$ here.

When $0 \le e < a + b$,

$$\phi(e) = \begin{cases} k_1 e, \ 0 \le e < b; \\ \frac{1}{a}(k_2 - k_1)e^2 - \frac{b}{a}(k_2 - k_1)e \\ \quad + k_1 e, \ b \le e < a + b. \end{cases} \tag{8}$$

$$\dot{\phi}(e) = \begin{cases} k_1, \ 0 \le e < b; \\ \frac{2}{a}(k_2 - k_1) - \frac{b}{a}(k_2 - k_1) + k_1, \ b \le e < a + b. \end{cases} \tag{9}$$

We can observe that both k_1 and $-\frac{b}{a}(k_2 - k_1) + k_1$ are constants. Since

$$\dot{\phi}(e)\big|_{e=b^-} = \frac{b}{a}(k_2 - k_1) + k_1, \tag{10}$$

$$\dot{\phi}(e)\big|_{e=(a+b)^+} = \frac{b}{a}(k_2 - k_1) + 2k_2 - k_1, \tag{11}$$

the maximum and minimum of $\dot{\phi}(e)$ must be in the set

$$A_1 = \{k_1, \frac{b}{a}(k_2 - k_1) + k_1, \frac{b}{a}(k_2 - k_1) + 2k_2 - k_1\}. \tag{12}$$

When $a + b \le e < (n-1)a + (2n-3)b$, there exists an integer $1 < i < n$, such that

$$e \in [(i-1)a + (2i-3)b, ia + (2i-1)b). \tag{13}$$

We can get

$$\phi(e) = \begin{cases} k_i e, \ (i-1)a + (2i-3)b \le e < (i-1)a + (2i-1)b; \\ \frac{1}{a}(k_{i+1} - k_i)e^2 - \frac{(i-1)a + (2i-1)b}{a}(k_{i+1} - k_i)e \\ \quad + k_i e, \ (i-1)a + (2i-1)b \le e < ia + (2i-1)b. \end{cases} \tag{14}$$

$$\dot{\phi}(e) = \begin{cases} k_i, \ (i-1)a + (2i-3)b \le e < (i-1)a + (2i-1)b; \\ \frac{2}{a}(k_{i+1} - k_i)e - \frac{(i-1)a + (2i-1)b}{a}(k_{i+1} - k_i) \\ \quad + k_i, \ (i-1)a + (2i-1)b \le e < ia + (2i-1)b. \end{cases} \tag{15}$$

Because

$$\dot{\phi}(e)\,|_{e=[(i-1)a+(2i-1)b]^-} = \frac{(i-1)a + (2i-1)b}{a}(k_{i+1} - k_i) + k_i, \tag{16}$$

$$\dot{\phi}(e)\,|_{e=[ia+(2i-1)b]^-} = \frac{(i-1)a + (2i-1)b}{a}(k_{i+1} - k_i) + 2k_{i+1} - k_i. \tag{17}$$

Then the maximum and minimum of $\dot{\phi}(e)$ must be in A_i.

When $(n-1)a + (2n-3)b \le e$, then $\phi(e) = k_n e$, and there is

$$\dot{\phi}(e) = k_n. \tag{18}$$

From the above analysis, it is apparent that the maximum and minimum of $\dot{\phi}(e)$ in the domain R are in the set $A=\bigcup_{i=1}^{n} A_i$. Thus, $\phi(e)\in M_{[k_{\min},k_{\max}]}$. ■

3. A Case Study of the Influences of Reference Inputs

Consider a simple T-S fuzzy control system as shown in Fig. 1, where

$$G(s)=\frac{1}{(0.2s+1)(0.1s+1)(0.05s+1)}. \tag{19}$$

The FLC consists of the following two rules:

$$\text{If } e \text{ is } M_1\text{, then } u_1=8.2e, \tag{20}$$

$$\text{If } e \text{ is } M_2\text{, then } u_2=10e, \tag{21}$$

where M_1 and M_2 are the fuzzy sets, whose membership functions μ_1 and μ_2 are given in (22) and (23) as well as in Fig. 3, where $a=\frac{2}{3}$, and $b=\frac{1}{3}$.

$$\mu_1(e)=\begin{cases} 0, e<-1; \\ \frac{3}{2}(e+1), -1\le e<-\frac{1}{3}; \\ 1, -\frac{1}{3}\le e<\frac{1}{3}; \\ -\frac{3}{2}(e-1), \frac{1}{3}\le e<1; \\ 0, e\ge 1. \end{cases} \tag{22}$$

$$\mu_2(e)=\begin{cases} 1, e<-1; \\ -\frac{3}{2}(e+\frac{1}{3}), -1\le e<-\frac{1}{3}; \\ 0, -\frac{1}{3}\le e<\frac{1}{3}; \\ \frac{3}{2}(e-\frac{1}{3}), \frac{1}{3}\le e<1; \\ 1, e\ge 1. \end{cases} \tag{23}$$

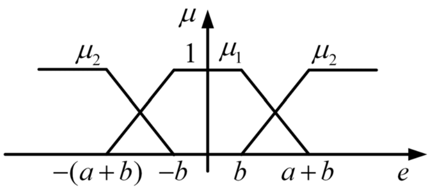

Fig. 3. The membership functions of M_1 and M_2.

This T-S fuzzy control system can be represented by the state-space form below:

$$\begin{cases} \dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}\varphi(e), \\ y=\boldsymbol{c}\boldsymbol{x}, \\ e=r-y. \end{cases} \tag{24}$$

where,

$$\boldsymbol{A}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1000 & -350 & -35 \end{bmatrix},\ \boldsymbol{b}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \boldsymbol{c}^T=\begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

If $r=0$, then $\boldsymbol{x}_e=0$ is the equilibrium point of the above T-S fuzzy control system. We can conclude that the equilibrium is globally asymptotically stable by using the circle criterion [20]. However, it is not guaranteed that our T-S fuzzy control system is also Lyapunov stable, when r is chosen to be other constant values. For example, the output in Fig. 4, as $r=8.4$, apparently demonstrates that the T-S fuzzy control system is not asymptotically stable.

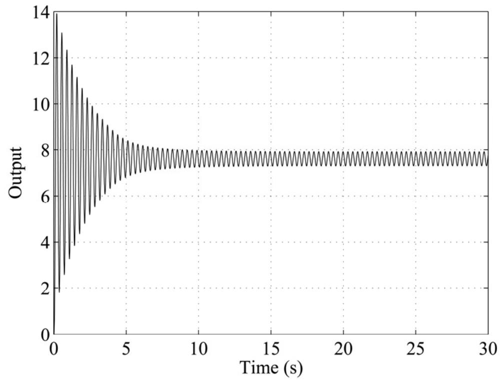

Fig. 4. The output of T-S fuzzy control system with reference input $r=8.4$.

Moreover, we can show that the equilibrium is even unstable. Actually, the equilibrium $\boldsymbol{x}_e\approx[0.0076\ \ 0\ \ 0]^T$ under the reference input $r=8.4$ can be derived by solving the following equation

$$\dot{\boldsymbol{x}}_e=\boldsymbol{A}\boldsymbol{x}_e+\boldsymbol{b}\varphi(r-\boldsymbol{c}\boldsymbol{x}_e)=0. \tag{26}$$

If the closed-loop system is asymptotically stable, $e_e=r-\boldsymbol{c}\boldsymbol{x}_e\approx 0.8$ is the steady state of the error signal. By linearizing the nonlinearity $\phi(e)$ at $e=0.8$, we have

$$\dot{\phi}(e)\Big|_{e=0.8}=5.4e+7.3=11.62. \tag{27}$$

It is easy to verify that the Nyquist plot of $G(s)$ encircles the point $(-\frac{1}{11.62},0)$ once, which implies that the linearization of this T-S fuzzy control system at the equilibrium point $\boldsymbol{x}_e$ is unstable, when $r=8.4$.

From the above case study, it can be concluded that the asymptotic stability of the T-S fuzzy control system (when $r = 0$) does not simply imply that the same system is asymptotically stable when r is other constant inputs. Although this example is illustrated by the T-S fuzzy control system, a Mamdani fuzzy control system can also be constructed to demonstrate the similar effects of the reference inputs.

4. Stability Analysis

In this section, we use the off-axis circle criterion to analyze the stability of the T-S fuzzy control system in Fig. 1. The main results are summarized in Theorem 2. We first briefly introduce some important conclusions of Ref. 23.

Theorem 1: Given the T-S fuzzy control system with the fuzzy logic rules (1) in Fig. 1, where $r = 0$, and $G(s)$ is the transfer function of the nominal linear plant. Denote $\phi(e)$ as the nonlinear mapping achieved by the T-S fuzzy controller and $\phi(e) \in M_{[k_{\min}, k_{\max}]}$. This system is globally asymptotically stable, if one of the following three conditions is satisfied:

1). If $0 = k_{\min} < k_{\max}$, $G(s)$ is Hurwitz, and the Nyquist plot of $G(j\omega)$ lies entirely to the right of a straight line passing through the point $(-\frac{1}{k_{\max}}, 0)$, as ω varies from 0 to $+\infty$;

2). If $0 = k_{\min} < k_{\max}$, the Nyquist plot of $G(j\omega)$ encircles the point $(-\frac{1}{k_{\max}}, 0)$ m times in the counterclockwise direction, as ω varies from $-\infty$ to $+\infty$, and lies outside an off-axis disk passing through the points $(-\frac{1}{k_{\min}}, 0)$ and $(-\frac{1}{k_{\max}}, 0)$, as ω varies from 0 to $+\infty$, where m is the number of the poles of $G(s)$ with positive real parts.

3). If $k_{\min} < 0 < k_{\max}$, $G(s)$ is Hurwitz, and the Nyquist plot of $G(j\omega)$ lies entirely inside an off-axis disk passing through the points $(-\frac{1}{k_{\min}}, 0)$ and $(-\frac{1}{k_{\max}}, 0)$, as ω varies from 0 to $+\infty$.

Proof. Refer to Ref. 23. ■

For the T-S fuzzy control system in Fig. 1, different from the circle criterion-based stability conditions, the following novel condition on the basis of the off-axis circle criterion can guarantee the global asymptotic stability, when $r = r_0$, $r_0 \in R$, if the closed-loop system is proven to be globally asymptotically stable by Theorem 1.

Theorem 2: Suppose the T-S fuzzy control system in Fig. 1 is proven to be globally asymptotically stable by Theorem 1. Then the T-S fuzzy control system is globally asymptotically stable, when $r = r_0$, $r_0 \in R$.

Proof. The state space equations of the T-S fuzzy control system are given as follows:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\varphi(e), \\ y = \boldsymbol{c}\boldsymbol{x}, \\ e = r - y. \end{cases} \tag{28}$$

Assume that $\boldsymbol{x}_e$ and e_e are the equilibrium points corresponding to the given reference input $r = r_0$, $r_0 \in R$. With (28), we have

$$\begin{cases} \dot{\boldsymbol{x}}_e = \boldsymbol{A}\boldsymbol{x}_e + \boldsymbol{b}\varphi(e_e) = 0, \\ y_e = \boldsymbol{c}\boldsymbol{x}_e, \\ e_e = r - y_e. \end{cases} \tag{29}$$

Subtracting (29) from (28) yields

$$\begin{cases} \dot{\overline{x}} = \boldsymbol{A}\overline{\boldsymbol{x}} + \boldsymbol{b}\overline{\varphi}(\overline{e}), \\ \overline{y} = \boldsymbol{c}\overline{\boldsymbol{x}}, \\ \overline{e} = \overline{r} - \overline{y}. \end{cases} \tag{30}$$

where $\overline{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_e$, $\overline{e} = e - e_3$, $\overline{y} = y - y_e$, $\overline{r} = 0$, and $\overline{\varphi}(\overline{e}) = \varphi(\overline{e} + e_e) - \varphi(e_e)$, whose nonlinearity is shown in Fig. 5. Here, we shift the equilibrium to the origin point, and as a consequence, the reference input can be considered to be zero. From Fig. 5, it is obvious that $\phi(\cdot)$ and $\overline{\phi}(\cdot)$ also belongs to $M_{[k_{\min}, k_{\max}]}$.

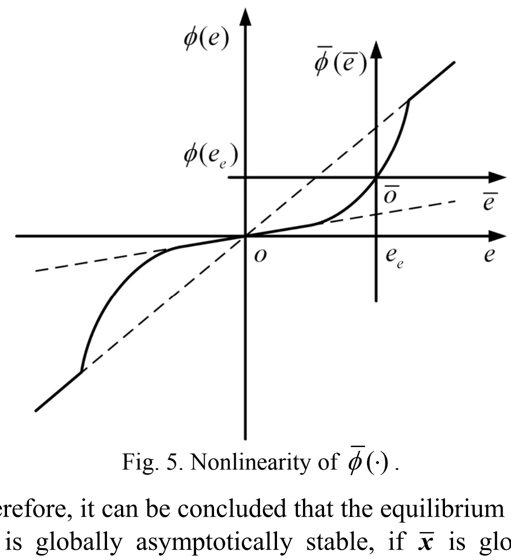

Fig. 5. Nonlinearity of $\overline{\phi}(\cdot)$.

Therefore, it can be concluded that the equilibrium point $\boldsymbol{x}_e$ is globally asymptotically stable, if $\overline{\boldsymbol{x}}$ is globally asymptotically stable. In other words, $\boldsymbol{x}_e$ and $\overline{\boldsymbol{x}}$ have the same stability property. Since system (30) is globally asymptotically stable by hypothesis, the T-S fuzzy control system in Fig. 5 is also globally asymptotically stable, when $r = r_0$, $r_0 \in R$. ■

By Theorem 2, the global asymptotic stability of different equilibriums is guaranteed by that of the equilibrium at the origin. The off-axis circle criterion focuses on the global property of the nonlinearity, while the circle criterion and Popov criterion can only handle the local property. Note that the off-axis circle criterion is applicable to those cases, where the prior knowledge of the nonlinearity is known beforehand. As a matter of fact, the stability analysis of different equilibriums can be unified by the off-axis circle criterion.

5. Simulations

Example 1. For the T-S fuzzy control system in Section 3, two proportional controllers $k_1 = 1.73$ and $k_2 = 3.16$ can be obtained by using the Bode plot. We construct a T-S fuzzy controller as follows:

$$\text{If } e \text{ is } M_1, \text{ then } u_1 = 1.73e, \tag{31}$$

$$\text{If } e \text{ is } M_2, \text{ then } u_1 = 3.16e. \tag{32}$$

The membership functions are shown in Fig. 3, where $a = 0.6$ and $b = 0.4$. By Lemma 1, we have $\phi(e) \in M_{[1.73,\,5.544]}$. The Nyquist plot of $G(j\omega)$ with the disk passing through points $(-\frac{1}{1.73}, 0)$ and $(-\frac{1}{5.544}, 0)$ is shown in Fig. 6, where the dashed line is the Nyquist plot of $G(j\omega)$, as ω varies from $-\infty$ to 0.

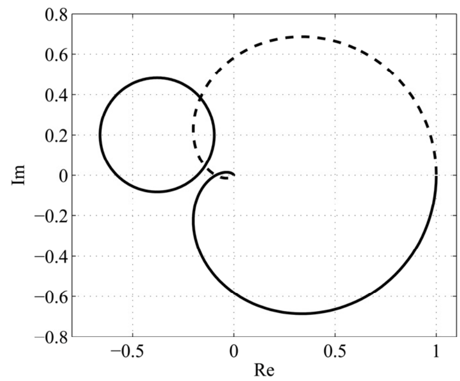

Fig. 6. Nyquist plot of $G(j\omega)$ and off-axis disk passing through points $(-\frac{1}{1.73}, 0)$ and $(-\frac{1}{5.544}, 0)$.

Apparently, with Theorem 2, the T-S fuzzy control system is globally asymptotically stable for any constant reference input. As an illustrative example, Fig. 7 shows the responses of the closed-loop system, when $r = 1$, $r = 5$, and $r = 10$.

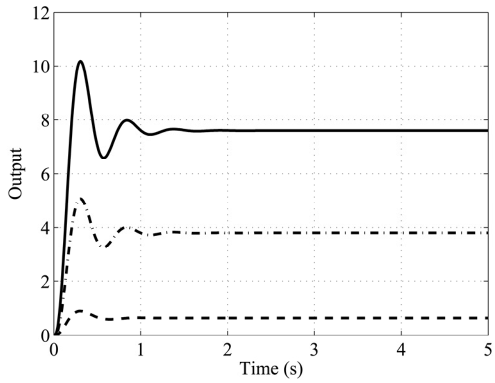

Fig. 7. Responses of T-S fuzzy control system, when $r = 1$ (dash line), $r = 5$ (dash-dot line), and $r = 10$ (solid line).

Example 2 [24]. Construct an FLC for the T-S fuzzy control system in Fig. 1, where the plant is unstable, and its transfer function is as follows:

$$G(s) = \frac{2s+1}{s(s-1)(0.5s+1)}. \tag{33}$$

The T-S fuzzy controller is built as in (31) and (32), where $k_1 = 1.25$ and $k_2 = 1.6$. The membership functions of this fuzzy controller are the same as that of Example 1. By Lemma 1, we know that $\phi(e)$ belongs to $M_{[1.25, 2.184]}$, and we can find an off-axis disk passing through points $(-\frac{1}{1.25}, 0)$ and $(-\frac{1}{2.184}, 0)$ without intersecting or being tangent with the Nyquist plot of $G(j\omega)$, as ω varies from 0 to $+\infty$ (in Fig. 8). The Nyquist plot also encircles point $(-\frac{1}{1.25}, 0)$, as ω varies from $-\infty$ to $+\infty$.

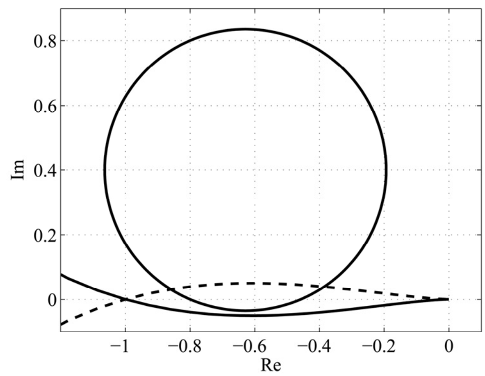

Fig. 8. Nyquist plot of $G(j\omega)$ and off-axis disk passing through points $(-\frac{1}{1.25}, 0)$ and $(-\frac{1}{2.184}, 0)$.

On the basis of Theorem 2, we conclude that our T-S fuzzy control system is asymptotically stable with $k_1 = 1.25$ and $k_2 = 1.6$.

Example 3: In this example, the well-known inverted pendulum problem is used to demonstrate how Theorem 2 can be applied to a nonlinear control system, whose model is given as follows:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{g \sin x_1 - amlx_2^2 \sin 2x_1 / 2 - a \cos x_1 \cdot u}{4l/3 - aml \cos^2 x_1}. \end{aligned} \tag{34}$$

where x_1 represents the vertical angle of the pendulum, x_2 is the angular velocity, $g = 9.8\, m/s^2$ is the gravity constant, m is the pendulum mass, M is the cart mass, l is the half length of the pendulum, u is the force exerted on the cart, and $a = \frac{1}{m+M}$. Here, we assume that $M = 8.0\, kg$, $m = 2.0\, kg$, and $l = 0.5\, m$.

When x_1 and x_2 are about zero, we obtain

$$\begin{aligned} \sin x_1 &\approx x_1, \\ \cos x_1 &\approx 1, \\ x_2^2(t) &\approx 0. \end{aligned} \tag{35}$$

By substituting (35) into (34), we linearize the nonlinear model of the inverted pendulum as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{gx_1 - au}{4l/3 - aml}. \end{aligned} \tag{36}$$

When x_1 is about $\frac{\pi}{8}$, and x_2 is about zero, we have:

$$\begin{aligned} \sin x_1 &\approx \frac{8}{\pi} \sin\frac{\pi}{8} \cdot x_1, \\ \cos x_1 &\approx \cos(\frac{\pi}{8}), \\ x_2^2(t) &\approx 0. \end{aligned} \tag{37}$$

The linearized model at $x_1 = \frac{\pi}{8}$ is

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{g\frac{8}{\pi}\sin\frac{\pi}{8} \cdot x_1 - a\cos\frac{\pi}{8} \cdot u}{\frac{4l}{3} - aml\cos^2\frac{\pi}{8}}. \end{aligned} \tag{38}$$

When x_1 is about $-\frac{\pi}{8}$, and x_2 is about zero, the linearized model can be similarly obtained as in (38). Therefore, there are two linearized models. When x_1 and x_2 are both about zero, the transfer function is

$$G_1(s) = -\frac{3}{17s^2 - 294}. \tag{39}$$

When x_1 is about $\pm\frac{\pi}{8}$, and x_2 is about zero, the transfer function is

$$G_2(s) = -\frac{0.1589}{s^2 - 16.428}. \tag{40}$$

To stabilize the inverted pendulum at $x_1 \approx 0$, an appropriate linear compensator is derived by the Bode plot as follows:

$$C_1(s) = -k_1 C(s) = \frac{-138(\frac{1}{2.7}s + 1)}{\frac{1}{80}s + 1}. \tag{41}$$

When x_1 is about $\pm\frac{\pi}{8}$, the compensator is acquired in the same way, which is

$$C_2(s) = -k_2 C(s) = \frac{-150(\frac{1}{2.7}s + 1)}{\frac{1}{80}s + 1}. \tag{42}$$

If we embed the dynamics of the two compensators into the inverted pendulum, we can acquire a generalized plant as

$$G(s) = G_1(s)C(s) = \frac{3(\frac{1}{2.7}s + 1)}{(17s^2 - 294)(\frac{1}{80}s + 1)}. \tag{43}$$

On the basis of the gains of the linear compensators given above, a T-S fuzzy controller is constructed as

If x_1 is about zero, then $u_1 = 138e$, (44)

If x_1 is about $\pm\frac{\pi}{8}$, then $u_2 = 150e$. (45)

The corresponding membership functions are shown in Fig. 3, where $a = \frac{3}{32}\pi$, and $b = \frac{\pi}{32}$. By Lemma 1, we have $\varphi(e) \in M_{[138,166]}$. The Nyquist plot of $G(j\omega)$ is given in Fig. 9, from which we can conclude that this inverted pendulum system is indeed asymptotically stable. For example, when $x_1(0) = 8^\circ$, the stable system response is illustrated in Fig. 10.

6. Conclusions

In this paper, we present a sufficient stability condition for a type of T-S fuzzy control systems by using the off-axis circle criterion. Different from other frequency domain-based approaches, it can guarantee the global

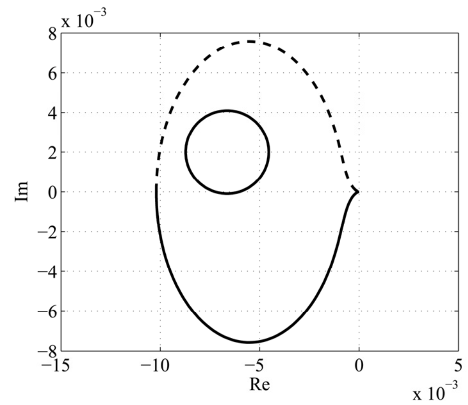

Fig. 9. Nyquist plot of $G(j\omega)$ and off-axis disk passing through points $(-\frac{1}{138},0)$ and $(-\frac{1}{166},0)$.

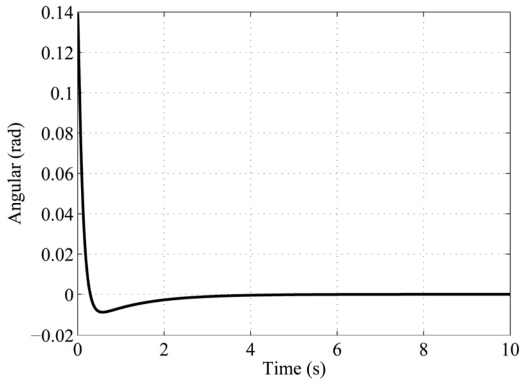

Fig. 10. The angular trajectory of inverted pendulum with initial angle of $8^\circ \approx 0.1396\ rad$.

asymptotic stability in case of various constant reference inputs. The proposed method has also a simple graphical interpretation. Three simulation examples have been given to demonstrate its efficiency. In the future work, we are going to generalize our stability analysis technique to the fuzzy control systems with interval plants.

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