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On the Lattice of Convex Sub-lattices of a Lattice

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Abstract. In his classical work imply that [1], G. Gratzer had asked that for what equational classes $\mathbf{K}$, $L \in \mathbf{K}$ and $Csub(L) \cong Csub(L_1)$ imply that $L_1 \in \mathbf{K}$, (see Problem 1.10 in [1]).In the present note, a necessary and sufficient condition for such classes will be given.

1. Introduction

G. Gratzer asked in [1] which equational classes $\mathbf{K}$, $L \in \mathbf{K}$ and $Csub(L) \cong Csub(L_1)$ imply that $L_1 \in \mathbf{K}$ (see Problem 1.10 in [1]). In the present note a necessary and sufficient condition for a class to have this property will be given and the following result will be proved.

For concepts not defined here one may refer to [2] and [3]. First it will be shown the ordering defined in [3] and [4] on the lattice of convex sub-lattices can be described as the smallest ordering satisfying certain conditions.

In this present note we prove the following result:

Main Theorem. *An equational class* $\mathbf{K}$ *has the property*:

(*) $$L \in \mathbf{K},\quad Csub(L) \cong Csub(L_1) \Rightarrow L_1 \in \mathbf{K}$$

if and only if $\mathbf{K}$ *is self-dual*.

Definition 1. An equational class $\mathbf{K}$ is said to be self-dual, if $\varepsilon \in Id(\mathbf{K}) \;\Rightarrow\; \varepsilon^c \in Id(\mathbf{K})$,

where ε^c is the dual identity of ε .

An easy consequence of the definition is:

Lemma 1. *Equational class* $\mathbf{K}$ *is self-dual if and only if* $L \in \mathbf{K} \;\Rightarrow\; L^c \in \mathbf{K}$,

where L^c *is the dual lattice of* L .

2.Definitions and Lemmas

In the following, symbols $\vee$ and $\wedge$ are used to denote the join and the meet operations in $Csub(L)$, the lattice of all convex sub-lattices of a lattice L ,respectively.

The necessary part of the Main Theorem is almost trivial, namely we have

Theorem 1. *If equational class* $\mathbf{K}$ *has the property* (*), *then* $\mathbf{K}$ *is self-dual*.

Proof. For any lattice L , since

$$A \in Csub(L) \;\Rightarrow\; A \in Csub(L^c)\,,$$

we have $(Csub(L),\subseteq) \cong (Csub(L^c),\subseteq)$ under identical mapping. Thus, if $\mathbf{K}$ has the property (*) and $L \in \mathbf{K}$, we deduce at once $L^c \in \mathbf{K}$ and $\mathbf{K}$ is self-dual. □

For the sufficient part, we begin with considering $\{\mathbf{0,1}\}$ -1attices.The following lemma is useful.

Lemma 2. *If* L *is* $\{\mathbf{0,1}\}$ *-lattice and* $Csub(L) \cong Csub(L_1)$ *, then* L_1 *is* $\{\mathbf{0,1}\}$ *-lattice.*

Proof. Let φ be an isomorphism of $Csub(L)$ onto $Csub(L_1)$. From the atomicity of the lattice of all convex sub-lattice of a lattice, $\varphi(\{a\})=\{a'\}$ for every $a\in L$, $\exists a'\in L_1$. Suppose that

$$\varphi(\{0\})=\{p'\},\quad \varphi(\{1\})=\{q\},(\exists p,q\in L_1),$$

Since $[p\wedge q,p\vee q]=\{p\}\vee\{q\}=\varphi(\{0\}\vee\{1\})=\varphi(L)=L_1,$ $p\wedge q, p\vee q$ are the least and greatest elements of L_1, respectively. □

Theorem 2. *Let* $\mathbf{K}$ *be a self-dual equational class, and* $L\in\mathbf{K}$ *be a* $\mathbf{\{0,1\}}$ *-lattice. If* $Csub(L)\cong Csub(L_1)$, *then* $L_1\in\mathbf{K}$.

Proof. Let φ be an isomorphism of $(Csub(L),\subseteq)$ onto $(Csub(L_1),\subseteq)$. Since $\varphi(\{a\})=\{a'\}$ for every $a\in L$, $\exists a'\in L_1$. We can define a map $f:L\to L_1$ by $f(a)=a'$. Obviously, f is bijective. The proof of the theorem is performed for three cases as follows.

Case 1. Suppose that $\varphi(\{0\})=\{0'\}$, where $0'$ and $1'$ are respectively the least and greatest elements of L and L_1 ensured by Lemma 2. We will prove that $L\cong L_1$ in this case.

Let $a,b\in L, a\le b$. Since $(f(a)]=\{0'\}\vee\{f(a)\}=\phi(\{0\})\vee\phi(\{a\})=\phi(\{0\}\vee\{a\})=\psi((a])\subseteq\phi((b])=(f(b)]$

we have $f(a)\le f(b)$, and f is order-preserving . Symmetrically, f^{-1} is also order-preserving and so f is a lattice-isomorphism.

The same is true for the case $\varphi(\{1\})=\{1'\}$.

Case 2. Suppose that $\varphi(\{0\})=\{1'\}$ or $\varphi(\{1\})=\{0'\}$. As in the case 1 $a\le b\Longleftrightarrow f(a)\ge f(b), a,b\in L$,

and then $L\cong(L_1)^c$.

Case 3. Suppose that $\{f(0),f(1)\}\cap\{0',1'\}=\varnothing$. We will prove $L_1\in\mathbf{K}$.

Suppose that $\varphi(\{0\})=\{p'\},\varphi(\{1\})=\{q'\}, \varphi(\{p\})=\{0'\}$ and $\varphi(\{q\})=\{1'\}$,that is, $f(0)=p', f(1)=q'$, $f(p)=0'$ and $f(q)=1'$. Since $p'\wedge q'\in[0',p']\cap[0',q']$ and

$$\begin{aligned}[0',p']\cap[0',q']&=(\{0'\}\vee\{p'\})\cap(\{0'\}\vee\{q'\}) =(\varphi(\{p\})\vee\varphi(\{0\}))\cap(\varphi(\{p\})\vee\varphi(\{1\}))\\ &=\varphi((\{p\}\vee\{0\})\cap(\{p\}\cap\{1\})) =\varphi([0,p]\cap[p,1]) =\varphi(\{p\}) =\{0'\},\end{aligned}$$

we have $p'\wedge q'=0'$. Similarly, since

$$[0,p'\vee q']=[0',p']\vee[0',q'] =\varphi((\{p\}\vee\{0\}\vee\{p\}\vee\{1\}) =\varphi(L)=L_1.$$

we have $p'\vee q'=1'$.

Symmetrically, we have also $p\vee q=1,$ $p\wedge q=0$.

On the other hand, since φ is isomorphic, we have $((q]]_{Csub(L)}=Csub((q]),([p')]_{Csub(L_1)}=Csub([p'))$.

Since $\varphi((q])=[p'),((q]]_{Csub(L)}\cong([p')]_{Csub(L_1)}$, and so $Csub((q])\cong Csub([p'))$ under φ. Thus for lattices $(q]$ and $[p')$, the condition stated in the case 1 are valid, and we conclude that $(q']=[p')$ under f .

Analogously, we can show that under f we have $(p]=(p']^c,[p)=(q']$ and $[q)'=[q')^c$. Now we proceed to show that $L\cong(p]\times(q]$. In fact, for $r\in L$, let $f(r)=r'$. Since

$$\begin{aligned}[p\wedge r,p\vee r]\cap[q\wedge r,q\vee r]&=(\{p\}\vee\{r\})\cap(\{q\}\vee\{r\})\\ &=\varphi^{-1}((\{0'\}\vee\{r'\})\cap(\{1'\}\vee\{r'\})) =\varphi^{-1}([0',r']\cap[r',1'])=\{r\} .\end{aligned}$$

And $(p\wedge r)\vee(q\wedge r)\in[p\wedge r,p\vee r]\cap[q\wedge r,q\vee r],$

we conclude that $r{=}(p\wedge r)\vee(q\wedge r)$, and every element $r\in L$ can be represented by a pair of elements $(p\wedge r,\ q\wedge r)$. To show that this representation is also unique, suppose that $r=P_0\vee q_0,P_0\in(p]$ and $q_0\in(q]$. From $p_0\le r$ and $q_0\le r$, we deduce that $p_0\le p\vee r\le p$ and $q_0\le q\vee r\le q$,so $f(q_0)\in[p'), f(q\wedge r)\in[p')$ and $f(p\wedge r)\in(q']$ .Therefore

$$f(q_0) \geq f(p_0), f(q \wedge r) \geq f(p \wedge r) \ .$$

Thus we have

$$[f(p_0), f(q_0)] = \varphi(\{p_0\} \vee \{q_0\}) = \varphi([0,r]) = \varphi(\{p \wedge r\} \vee \{q \wedge r\})$$
$$= \{f(p \wedge r)\} \vee \{f(q \wedge r)\} = [f(p \wedge r), f(q \wedge r)],$$

and so $f(p_0) = f(p \wedge r)$, $f(q_0) = f(q \wedge r)$,that is, $p_0 = p \wedge r$ and $q_0 = q \wedge r$, and therefore the representation is unique

Now for any tow elements, $r, s \in L$, if $r \leq s$,then $p \wedge r \leq p \wedge s$ and $q \wedge r \leq q \wedge s$ .Conversely, if $p \wedge r \leq p \wedge s$ and $q \wedge r \leq q \wedge s$, $r = (p \wedge r) \vee (q \wedge r) \leq (p \wedge s) \vee (q \wedge s) = s$. This means that

$$r \leq s \Leftrightarrow p \wedge r \leq p \wedge s \text{ and } q \wedge r \leq q \wedge s, r, s \in L,$$

and therefore the map $\theta : r \mapsto (p \wedge r) \vee (q \wedge r)$ is an isomorphism of L onto $(p] \times (q]$.

Similarly, we have $L_1 \cong (p'] \times (q']$.

If $L \in \mathbf{K}$, then $(q], [p) \in \mathbf{K}$, so that $(q'], (p']^c \in \mathbf{K}$. By assumption, $\mathbf{K}$ is self-dual, we have $(p'] \in \mathbf{K}$ and is follows that $L_1 \in \mathbf{K}$ and the proof is now complete. □ .

3. The Proof of Main Theorem

From Theorem 1 and Theorem 2, we know that Main Theorem is true for the case of bounded lattices. In the following, we are going to deal with the general case.

Lemma 3. *For any lattice* L, *identity* $p = q$ *holds in* L *if and only if it holds in every interval of* L.

Now we are in a position to establish:

Theorem 3. *Let* $\mathbf{K}$ *be a self-dual equational class and* $Csub(L) \cong Csub(L_1)$. *If* $L \in \mathbf{K}$, *then* $L_1 \in \mathbf{K}$.

Proof. Maps φ and f are the same as in the proof of Theorem 2. For any interval $[a', b'] \subseteq L_1$, let $f(a) = a'$, and $f(b) = b'$. Since

$$[a \wedge b, a \vee b] = \{a\} \vee \{b\} = \varphi^{-1}(\{a'\} \vee \{b'\}) = \varphi^{-1}([a', b']),$$

We have: $Csub([a \wedge b, a \vee b) \cong ([a \wedge b, a \vee b]]_{Csub(L)} \cong ([a', b')]_{Csub(L_1)} \cong Csub([a', b'])$

under φ, that is

$$\varphi|_{Csub([a \wedge b, a \vee b])} : Csub([a \wedge b, a \vee b)) \mapsto Csub([a', b'])$$

is an isomorphism . Without loss of generality, we can assume that $a \wedge b < a, b < a \vee b$ and let $p' = f(a \wedge b)$, $q' = f(a \vee b)$. Since $\varphi(\{a \wedge b\}), \varphi(a \vee b) \in Csub([a', b')), p', q' \in [a', b']$. As in the case 3 of the proof of Theorem 2, we have

$$p' \vee q' = b', \ \ p' \wedge q' = a', [a \wedge b, a \vee b] \cong [a \vee b, b] \times [a \wedge b, a], [a', b'] \cong [a', p'] \times [a', q'],$$

$$[a \wedge b, b] \cong [p', b'], \ [a \wedge b, a] \cong [a', p']^c, \ [a, a \vee b] \cong [a', q'], \ [b, a \vee b] \cong [q', b']^c.$$

For any identity $\varepsilon \in Id(\mathbf{K})$, both ε and ε^c hold in $[a, a \vee b]$ and $[a \wedge b, a]$ since they hold in L , and then they hold in $[a', q']$ and $[a', p']^c$. It is follows that ε holds in $[a', q']$ and $[a', p']$, so in $[a', p']^c$. Our proof is complete by Lemma 3. □

The Main Theorem is consequence of Theorem 1 and Theorem 3

4. Some Corollaries

By the Proof of Theorem 2, we have:

Corollary 1. *Let* L *be a* $\{\mathbf{0,1}\}$ *-lattice. If* $Csub(L) \cong Csub(L_1)$, *then there exists* $A, B \in Csub(L)$ such that $L \cong A \times B$ *and* $L_1 \cong A \times B$ *or* $A^c \times B$ *or* $A \times B^c$ *or* $A^c \times B^c$.

By the Theorem 3,we have:

Corollary 2[5]**.** *Let* L *be a bounded distribution lattice, and* $Csub(L) \cong Csub(L_1)$, *then* L_1 *is also a bounded distribution lattice.*

Corollary 3[6]**.** *Let* L *be a bounded semi-modular lattice, and* $Csub(L) \cong Csub(L_1)$, *then* L_1 *is also a bounded semi-modular lattice.*

5. An Interesting Application

As an interesting application we prove:

Theorem 4. *Let* L *be a* $\{\mathbf{0,1}\}$ *-lattice,* L_1 *be a lattice. Then the following conditions are equivalent:*

1) $Csub(L) \cong Csub(L_1)$ *implies that* $L \cong L_1$,

2) *if* $L = L_1 \times L_2 \times \cdots \times L_m$, *Then* $L_k \cong (L_k)^c, (k = 1, 2, \cdots, m)$.

Proof. 1) $\Rightarrow$ 2) If $L = L_1 \times L_2 \times \cdots \times L_m$, since $Csub(L) \cong Csub(L_1 \times \cdots \times L_{k-1} \times L_k{}^c \times L_{k+1} \times \cdots \times L_m)$,then $L = L_1 \times L_2 \times \cdots \times L_m \cong L_1 \times \cdots \times L_{k-1} \times L_k{}^c \times L_{k+1} \times \cdots \times L_m$.

Thus $L_k \cong (L_k)^c, (k = 1, 2, \cdots, m)$.

2) $\Rightarrow$ 1) Since $Csub(L) \cong Csub(L_1)$ and L *is a* $\{\mathbf{0,1}\}$ -lattice, by Corollary 1, we have that there exists $A, B \in Csub(L)$ such that $L \cong A \times B$ *and* $L_1 \cong A \times B$ *or* $A^c \times B$ *or* $A \times B^c$ *or* $A^c \times B^c$.

By 2),we have: $A \cong A^c$, $B \cong B^c$ .Then $A \times B \cong A \times B^c \cong A^c \times B \cong A^c \times B^c$,therefore $L \cong L_1$,and the proof is complete. □

Corollary 6. *Let* $L_i (i = 1, 2, \cdots, m)$ *be a china, and* $Csub(L) \cong Csub(L_1 \times L_2 \times \cdots \times L_m)$, *then* $L \cong L_1 \times L_2 \times \cdots \times L_m$.

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