

## Research on Structural Random Response Based on C Type Gram-Charlier

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**Abstract.** The main research content is a quantitative description of the structural uncertainty problem (Quantification Uncertainty) based on the finite element theory. For the first time, reduction integration method is used to decompose multi-dimensional random response function, the multi-dimensional function is transformed into the combination form of multiple one-dimensional random response function, so the multi-dimensional statistical moments of structural random response expressions can be expressed the sum of one-dimensional function integration; Gaussian integration formula is used to compute one-dimensional random response statistical moments, after the multi-dimensional random structural response statistical moments is gained, C type Gram-Charlier (CGC) Series expansion method is used to approximate the probability density function of random structural response. There is an example about car in this paper, it has two random variables that follow normal distribution and lognormal distribution, respectively, then the car's amplitude is computed, the results are compared with Monte Carlo method. CGC and MCS can match well, the tail of the random response probability distribution can also give an accurate estimate; the results illustrate that the CGC is suitable for normal distributions and lognormal distributions; The total number of computational times of CGC method are far less than the Monte Carlo method, the computation efficiency is much higher than Monte Carlo method. So CGC method in this paper is simple and reasonable.

### Introduction

With the rapid development of science and technology level, modern engineering structures exhibit complex and diversified trend, when analyzing the actual structural engineering, there are different degrees of error and uncertainty in complex structures, the uncertainty comes from 5 cases [1-3]: (1)material parameters; (2)the structure size parameters; (3)external load in service period; (4)initial and boundary conditions; (5)computation model. Usually these errors or uncertainties may be a little, but if they accumulated together it might generate a greater and unexpected deviations, so the role of uncertainties in the structure can not be ignored. In structural analysis considering the impact of uncertainties on the structural response has important theoretical signification and practical value [4].

Stochastic finite element method (Stochastic FEM) is also named probabilistic finite element method, this method is commonly used in engineering structural uncertainty analysis:

**Monte Carlo (MCS) Method.** MCS is the most simple method based on samples. MCS is very simple, the simulation times are used more, the calculation accuracy of the results is gradually higher and close to the theoretical solution. But the rate of convergence is very low. MCS requires a large amount of samples to get more accurate results, so it is not practical for engineering, it is usually used as an authentication method.

**Perturbation Stochastic Finite Element Method.** This method is not based on samples.

Normally this method can be performed up to 2th order expansion truncation, for higher order expansion truncation the method will be very complicated. The accuracy decreases with variation coefficient of random parameters and nonlinear degree of structural equation increases[5]. It generally applied in small-scale random input problems[6].

For the shortcomings of uncertainty analysis, dimension reduction integration method(DRM) and CGC series expansion method are used for structural analysis. Firstly using DRM to compute random statistical moments. Then CGC is used to approximate stochastic response probability density distribution (probability density function and probability distribution function, PDF and CDF). An example of nonlinear structure is analyzed and the results are compared with the MCS method. The results show that CGC method is suitable for normal distributions and lognormal distributions, it can obtain good accuracy, it also can reduce the consumption of computing resources and significantly improves the computational efficiency.

### Principle of Method

**CGC Series Expansion Method.** Compared with A type Gram-Charlier(AGC) method, CGC series can ensure that the PDF value is always positive. The definition of CGC series is:

$$f_Y(Y) = \frac{\exp\left[\sum_{j=1}^{\infty} \frac{1}{j} \gamma_j H_j(Y^*)\right]}{\int \exp\left[\sum_{j=1}^{\infty} \frac{1}{j} \gamma_j H_j(Y^*)\right] dY} \quad (1)$$

Here,  $\gamma_j$  is the  $j$  order series expansion coefficient,  $Y^* = (Y - \mu_Y) / \sigma_Y$  is equivalent standardization variables of  $Y$ ,  $H_j(Y^*)$  is  $j$  order Hermite polynomial. CGC series expansion coefficient  $\gamma$  can be calculated by the following linear equations:

$$AY = B \quad (2)$$

The computation of the matrix  $A$  is expressed as:

$$a_{uv} = \sum_{s=0}^{u+v-2} \frac{1}{s!} \Delta_{u-1, v-1, s} \psi_{Y^*, s} \quad (3)$$

Here

$$\Delta_{p, q, r} = \begin{cases} \prod_{\alpha=p, q, r} \frac{\Gamma(\alpha+1)}{\Gamma(\beta-\alpha+1)} & \text{if } p+q+r = \text{even}, \beta = (p+q+r)/2 \geq p, q, r \\ 0 & \text{elsewhere} \end{cases} \quad (4)$$

$B$  is a vectors, the computational expressions are as follows:

$$b_1 = 0, b_n = -(n-1)\psi_{Y^*, n-2}, n = 2, 3, \dots, m_{\max} \quad (5)$$

Hermite moments computation formula in CGC series:

$$\psi_{Y^*, n} = \int H_n(y^*) f_{Y^*}(y^*) dy = E[H_n(Y^*)] \quad (6)$$

So, we get CGC series expansion coefficient as following:

$$\gamma = A^{-1}B \quad (7)$$

**Dimension Reduction Integration Method.** When the researchers compute structural response statistical moment they often face multidimensional integration problems, Rahman and Xu proposed to solve this difficulty [7]. The DRM transforms the multidimensional integration problem into the sum of multiple one-dimensional functions, then calculating the response moments. The advantages of DRM are without solving derivative and matrix inversion of stochastic structural response function, so the application is simple and efficient.

DRM can be divided into three steps:

(1) Using the thought of additive decomposition, a multi-dimensional integration problem is transformed into a multiple one-dimensional integration problem.

(2) One-dimensional variable statistical moments approximate multidimensional stochastic structural response statistical moments.

(3) Applying numerical integration method based on moments—CGC is used to solve stochastic structure uncertainty problem, this method can obtain the structural response probability distribution [8].

DRM is used in this paper,  $N$ -dimensional response  $Y(X)$  is assumed and then decomposing it into the sum form of one-dimensional function:

$$Y(X) \cong \sum_{i=1}^N \hat{Y}_i(\mu_1, \dots, \mu_{i-1}, x_i, \mu_{i+1}, \dots, \mu_N) - (N-1)Y(\mu_1, \dots, \mu_N) \tag{8}$$

Here,  $\mu_i$  is mean of random variable  $x_i$ ,  $N$  is the amount of random variable in the response computation. The  $j$  order moment of  $N$ -dimensional function expressions as following:

$$m_Y^j = E[Y^j(X)] \cong E\left\{ \left[ \sum_{i=1}^n \hat{Y}_i(X) - (n-1)Y(\mu_1, \mu_2, \dots, \mu_n) \right]^j \right\} \tag{9}$$

Applying binomial theorem, statistical moments computation expression as following:

$$\begin{aligned} m_Y^j &= \sum_{l=0}^j \left\{ C_j^l E \left[ \sum_{i=1}^n \hat{Y}_i(X) \right]^l \left[ (1-n)Y_0 \right]^{j-l} \right\} \\ &= \sum_{l=0}^j \left\{ C_j^l E \left[ \sum_{i_1=0}^l C_l^{i_1} \sum_{i_2=0}^{l-i_1} C_{l-i_1}^{i_2} \dots \sum_{i_{n-1}=0}^{l-\sum_{k=1}^{n-2} i_k} C_{l-\sum_{k=1}^{n-2} i_k}^{i_{n-1}} \hat{Y}_1^{i_1} \hat{Y}_2^{i_2} \dots \hat{Y}_{n-1}^{i_{n-1}} \hat{Y}_n^{l-\sum_{k=1}^{n-1} i_k} \right]^l \left[ (1-n)Y_0 \right]^{j-l} \right\} \\ &= \sum_{l=0}^j \left\{ C_j^l \sum_{i_1=0}^l C_l^{i_1} \sum_{i_2=0}^{l-i_1} C_{l-i_1}^{i_2} \dots \sum_{i_{n-1}=0}^{l-\sum_{k=1}^{n-2} i_k} C_{l-\sum_{k=1}^{n-2} i_k}^{i_{n-1}} \left[ m(\hat{Y}_1) \right]^{i_1} \left[ m(\hat{Y}_2) \right]^{i_2} \dots \left[ m(\hat{Y}_{n-1}) \right]^{i_{n-1}} \left[ m(\hat{Y}_n) \right]^{l-\sum_{k=1}^{n-1} i_k} \left[ (1-n)Y_0 \right]^{j-l} \right\} \end{aligned} \tag{10}$$

When computing one-dimensional function statistical moments, the Gauss numerical integration formula which has high precision and good stability is used, the corresponding numerical integration formulas are in Table 1.

Table 1 Orthogonal Polynomials and Numerical Integration Formula

Distribution	PDF	Polynomials	Integration formula
Normal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	Hermite	$\sum_{k=1}^N \omega_k [h(\mu + x_i^k \sigma)]$
Lognormal	$f(x) = \frac{1}{\sqrt{2\pi}\zeta x} e^{-\frac{1}{2} \left( \frac{\ln x - \tilde{\mu}}{\zeta} \right)^2}$	Hermite	$\sum_{k=1}^N \omega_k \{h[e^{\tilde{\mu} + x_i^k \zeta}]\}$

### Numerical Example

In the example, there is a car driving on the rough roads, the quality of the car is  $m=490\text{kg}$ , vibration form of the automotive system is in Figure.1. Spring stiffness of the car is  $k=20\text{kg/cm}$ , the quality and deformation of the tire are negligible. Pavement shape is  $y(t) = Y \sin(2\pi x / L)$ , where  $y=4\text{cm}$ ,  $L=10\text{cm}$ . Horizontal velocity is  $v=36\text{km/h}$ , in this example the amplitude of the car is computed. The automotive system randomness is from quality 'm' and spring stiffness 'k'. Assuming the  $m$  obeys normal distribution that  $\mu=490\text{kg}$  and  $\sigma^2=0.05$ ,  $k$  follows lognormal distribution where  $\mu=20\text{kg/cm}$  and  $\sigma^2=0.01$ . The corresponding parameter and computation times are in table 1. Computation results are shown as below in the Figure2.

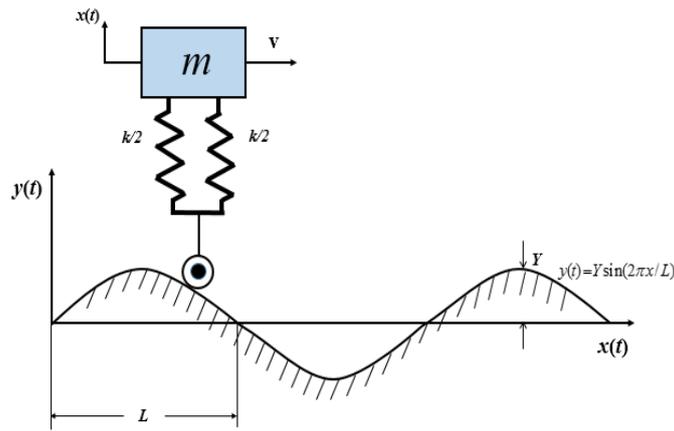


Figure.1. Automotive Systems structure

Table.2 Parameter Table of Random Variables

Parameter	Normal	Lognormal
Calculated Value		
variable	$k$	$m$
Mean	20	490
Variance	0.01	0.05
Runs of Monte Carlo	384	1920
MCS total runs	737280	
CGC total runs	11	

### Results

As can be seen from the Figure2, the results obtained from CGC method and Monte Carlo method (MCS) can be matched well.

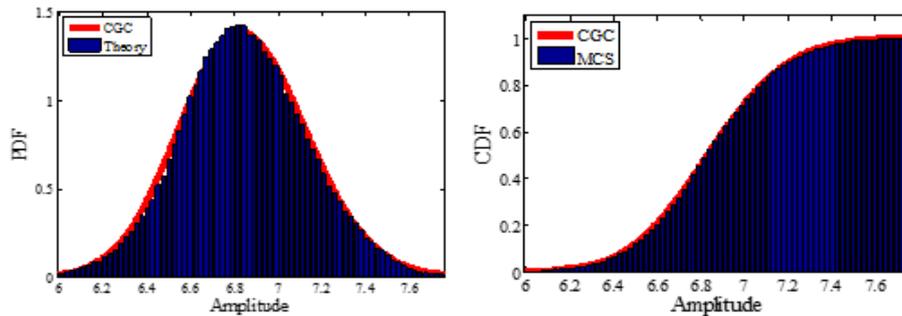


Figure.2. PDF and CDF of the car's amplitude with MCS and CGC method

## Conclusion

The computation results as mentioned above show that: (1)CGC is applicable for normal and lognormal distribution random independent variables; (2)CGC and MCS can match well from the figure 2, the tail of the random response probability distribution can also give an accurate estimate, so CGC can also estimate the small probability event; (3)MCS computation times are 6.7 million times as many times as CGC, it reduces the computation times, the computational cost is reduced efficiently.

## References

- [1] Chen Jianjun, Che J W, Sun H A, et.al. Probabilistic dynamic analysis of truss structures[J]. Structural Engineering and Mechanics, 2002,13(2):231-239.
- [2] Dai Jun, Chen Jianjun, Li Yonggong, et.al. Dynamic response optimization design for engineering structures based on reliability [J]. Applied Mathematics and Mechanics, 2003, 24(1):43-52.
- [3] Gao W, Chen J J, Ma H B, et.al. Dynamic response analysis of closed loop control system for intelligent truss structures based on probability[J]. Structural Engineering and Mechanics, 2003, 15(2):239-248.
- [4] Li Jinping. Research on Analysis Methods of Uncertain Temperature Fields and Structures. Xidian University, 2008.
- [5] Pandey M D. A numerical method for structural uncertainty response computation[J]. Science China Technological Sciences, 2011, 54(12):3347-3357.
- [6] Tang Tao, Zhou Tao. Recent developments in high order numerical methods for uncertainty quantification[J]. Science China Mathematics, 2015,58(7).
- [7] Rahman S, Xu H. A univariate dimension-reduction method for multi-dimensional integration in stochastic mechanics[J]. Probabilistic Engineering Mechanics,2004,19(4):393-408.
- [8] Zhang Kai, Gang L I. Reliability analysis based on the improved dimension reduction method[J]. Chinese Journal of Computational Mechanics, 2011,28(2):187-192.