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Consistency and stability in aggregation operators: An application to missing data problems

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Abstract

In this work we analyze the key issue of the relationship that should hold between the operators in a family $\{A_n\}$ of aggregation operators in order to understand they properly define a *consistent* whole. Here we extend some of the ideas about stability of a family of aggregation operators into a more general framework, formally defining the notions of $i-L$ and $j-R$ strict stability for families of aggregation operators. The notion of strict stability of order k is introduced as well. Finally, we also present an application of the strict stability conditions to deal with missing data problems in an information aggregation process. For this analysis, we have focused in the weighted mean family and the quasi-arithmetic weighted means families.

Keywords: Aggregation operators; stability; missing data; self-identity; fuzzy sets.

1. Introduction

An *aggregation operator* [1,6,3,7,8,10] is usually defined as a real function A_n such that, from n data items $x_1,\ldots,x_n$ in $[0,1]$, A_n produces an aggregated value $A_n(x_1,,x_n)$ in [0,1] (see e.g. [5]). This definition can be extended to consider the whole family of operators for any n instead of a single operator for a specific n. This has led to the current standard definition [5,12] of a *family of aggregation operators* (*FAO*) as a set $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$, providing instructions on how to aggregate collections of items of any dimension n. This sequence of aggregation functions $\{A_n\}_{n\in\mathbb{N}}$ is also called *extended aggregation functions* (*EAF*) by other authors [12,6]. In this work, we will deal with two different but related problems for *extended aggregation functions* or *families of aggregation operators*.

On one hand, let us remark that in practice, often some data can get lost, be deleted from or added to the information to be aggregated, and each time a cardinality change occurs a new aggregation operator A_m has to be used to aggregate the new collection of m elements. However, it is important to stress that a relation between $\{A_n\}$ and $\{A_m\}$ does not necessarily exist in a family of aggregation operators as defined in [5]. In this context, it seems natural to incorporate some properties to maintain the logical *consistency* between operators in a *FAO* when changes on the cardinality of the data occur, for which we need to be able to build up a definition of family of aggregation operators in terms of its logical consistency. That is, the operators that compose a *FAO* have to be somehow related, so the aggregation process remains *the same* throughout the possible changes in the dimension n of the data. Therefore, it seems appropriate to study properties giving sense to the sequences A_2, A_3, $A_4, \ldots$, because otherwise we may have only a bunch of disconnected operators. With this aim, in [15,16,11] the notion of *stability*, a kind of *consistency* based on the robustness of the aggregation process, was proposed and studied. In this sense, the notion of *stability* for a family of aggregation operators is inspired in continuity, though our approach focuses in the cardinality of the data rather than in the data itself, so it is possible to assure some robustness in the result of the aggregation process despite the possible cardinality changes. Particularly, let $A_n(x_1,\ldots,x_n)$ be the aggregated value of the n-dimensional data $x_1,\ldots,x_n$. Now, let us suppose that a new element x_{n+1} has to be aggregated. If x_{n+1} is close to the aggregation result $A_n(x_1,\ldots,x_n)$ of the n-dimensional data $x_1,\ldots,x_n$, then the result of aggregating these $n+1$ elements should not differ too much from the result of aggregating the previous n items. Following the idea of stability for any mathematical tool, if $|x_{n+1} - A_n(x_1,\ldots,x_n)|$ is small, then $|A_{n+1}(x_1,\ldots,x_n,x_{n+1}) - A_n(x_1,\ldots,x_n)|$ should be also small. It is important to note that if the family $\{A_n\}$ is not symmetric (i.e. there exist a n for which the aggregation operator A_n is not symmetric), then the position of the new data is relevant to the final output of the aggregation process. From this observation, in [15,16,11] some definitions of *stability* that extend the notion of self-identity defined in [18] were presented.

On the other hand, a problem that has not received too much attention is how to obtain an aggregation when some of the variables to be aggregated are missing. If the aggregation operator function A_n presents a clear definition for the case in which the dimension is lower, then this problem can be easily solved, though it is not always a trivial task. In this paper we will deal with the problem of missing data for some well-known families of aggregation operators by means of the consistency-based approach to aggregation families given by the notion of stability.

2. Consistency in families of aggregation operators.

As has been pointed out in the introduction, a *family of aggregation operators* (*FAO*) is a set of aggregation operators $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$, providing instructions on how to aggregate collections of items of any dimension n. In [12] it is shown that the operators of a family can be related by means of certain grouping properties. For example, continuity, symmetry or other well-known properties defined usually for aggregation functions A_n can be defined in a general way for a family of aggregation operators imposing that these properties have to be satisfied for all n. Nevertheless, this kind of properties does not guarantee any consistency in the aggregation process since they don't establish any constraint among the different aggregation functions.

Although few properties have been studied or defined to a *FAO* in general (see [16] for more details), in the aggregation operators' literature it is possible to find some properties for aggregation operators that can be understood as properties for the whole family establishing some relations among the different aggregation operators. Here we recall some of them.

An important notion that establishes relationships among operators of different dimension in a FAO is the notion of *recursivity*. Recursivity was introduced in [7] in the context of OWA operators functions. Following [7], in [1,8,10,4] recursivity of a FAO was studied in a more general way in connection with the consistency of an aggregation process. Note that recursivity guarantees a certain kind of consistency in a *FAO* $\{A_n\}$, since each function A_n is build taking into account the previous function A_{n-1}. Thus, the above mentioned situation, in which the different operators A_n have no relation among them, cannot hold for recursive operators.

Other properties that establish some conditions among the different members of the whole family are *decomposability* (see [5,6] for more details) or *bisymmetry* (see [5,6] for more details) among others.

Although these properties can be regarded as modeling a kind of consistency in a family of aggregation operators, it is more appropriate to say that they are more focused on the way in which it is possible to build the aggregation operator of dimension n from aggregation operators of lower dimensions than on a particular idea of consistency. On the other hand, pursuing the idea of consistency of a family of aggregation operators and based on the self-identity definition given by Yager in [18], in [15,16,11] the notion of *strict stability* of a FAO was defined in three different levels. Also, in [6] the self-identity property and its dual property were analyzed and studied in order to determine a consistent family of weights for different families of weighted aggregation operators.

The idea is simple: in a family of aggregation operators, A_n and A_{n+1} should be closely related, in the sense that if a new item has to be aggregated and such a new item is the result of the aggregation of the previous n items, then the result of the aggregation of these $n+1$ items should be close to the aggregation of the n previous ones. Otherwise, the aggregation operator A_{n+1} would differ too much from A_n, producing an *unstable* family $\{A_n\}_{n\in\mathbb{N}}$.

Taking into account that general FAOs are not necessarily symmetric, in [16] two possibilities (L- or left and R- or right stability) were analyzed concerning the definition of strict stability.

Definition 1. [15,16] Let $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ be a family of aggregation operators. Then, it is said that:

1. $\{A_n\}_n$ is an R-*strictly stable family* if $\forall\{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$, and $\forall n \geqslant 3$, $A_{n-1}(x_1,...,x_{n-1})$ coincides with
$$A_n(x_1,...,x_{n-1},A_{n-1}(x_1,...,x_{n-1})).$$
2. $\{A_n\}_n$ is an L-*strictly stable family* if $\forall\{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$, and $\forall n \geqslant 3$, $A_{n-1}(x_1,...,x_{n-1})$ coincides with
$$A_n(A_{n-1}(x_1,...,x_{n-1}),x_1,...,x_{n-1}).$$

Although the previous definitions can be relaxed from an asymptotic and probabilistic point of view (see [16]), in this work we are going to focus on the strict stability conditions just exposed.

3. On i-L and j-R stability

The previous definitions impose that the new datum appears in the last or in the first position. However, this assumption could be relaxed in order to allow it to appear in any position. In this way, the notion of j-L strict stability was introduced in [2], imposing that the new datum enters in the j-th position. And similarly, we can define the dual notion of i-R strictly stability by imposing that the new datum enters in the i-th position from the right. Obviously, the relaxed versions of j-L and i-R strict stability from an asymptotic and probabilistic point of view could be defined in a similar way.

Definition 2. Let $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ be a family of aggregation operators. Then, it is said that:

1. $\{A_n\}_n$ is a i-R-strictly stable family if $\forall\{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$, and $\forall n \geqslant 3$, $A_{n-1}(x_1,...,x_{n-1})$ coincides with
$$A_n(x_1,...,x_{n-i},A_{n-1}(x_1...,x_{n-1}),\dots,x_{n-1}).$$
2. $\{A_n\}_n$ is a j-L-strictly stable family if $\forall\{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$, and $\forall n \geqslant 3$, $A_{n-1}(x_1,...,x_{n-1})$ coincides with
$$A_n(x_1,...,x_{j-1},A_{n-1}(x_1,...,x_{n-1}),x_j,\dots,x_{n-1}).$$

Let us observe that the j-L and i-R strict stability notions previously defined are equivalent (for any j and/or i) when the FAO is symmetric. But, in general, it is very difficult that a non-symmetric FAO satisfies simultaneously more than one condition (see [16] for more details). In our opinion, the particular strict stability conditions that a general, non-symmetric FAO should satisfy have to take into account the structure of the data that have to be aggregated (and of course also the way in which this family is defined).

In a similar way as symmetric FAOs implicitly impose that the structure of the data has no effect in the aggregation result (since the order in which the information is aggregated is not relevant), non-symmetric families of aggregation operators makes the assumption that the data has an inherent structure, and thus the position of the data items in the aggregation process is relevant. Strict stability (among other properties) should also take into account that the data may present some structure (see [17,13] for more details). In the next section, we present some definitions of strict stability for non-symmetric FAOs that will be dependent on the structure of the data to be aggregated.

4. Strict stability of order k

The three different levels of stability presented in [16,15] (*strict*, *asymptotic* and *almost sure*) establish some constraints and relationships between the members of a FAO $\{A_n\}_n$ in order to guarantee some consistency and robustness in the aggregation process. Nevertheless, these conditions only focus on the relationships between two consecutive members, i.e. A_n and A_{n+1}. The notion of strict stability can be stated in a more general way by imposing conditions between the aggregation functions A_n an A_m when $|m-n| = k$. To this aim, we present the concept of strict stability of order k with respect to positions $r_1,\ldots,r_k$. As in the case of (order 1) strict stability, let us suppose that x is an n-dimensional vector to be aggregated by the operator A_n. Now, let us also suppose that k more data items, all of which coincide with the aggregation $A_n(x)$, are added to x. What should then be the aggregation of the $n+k$-dimensional vector that results from including the value $A_n(x)$ in the positions $r_1,\ldots,r_k$. In our opinion, in a consistent process the aggregation of the new vector should not differ too much from $A_n(x)$. In order to introduce this new concept, let us first introduce the following notation.

Given an specific $n \in N$, and given a sequence of k positions $r_1,\ldots,r_k$ with $r_1 < r_2 \ldots < r_k \leqslant n+k$, let us denote by $\xi^n_{r_1,r_2,\ldots,r_k}$ the function

$$\xi^n_{r_1,r_2,\ldots,r_k} : [0,1]^n \times [0,1] \longrightarrow [0,1]^{n+k}$$

which, for a given $x \in [0,1]^n$ and $\alpha \in [0,1]$, transforms the vector x into a vector in $[0,1]^{n+k}$ adding the value α in the positions $r_1,\ldots,r_k \leqslant n+k$. For example, let $x = (x_1,\ldots,x_6)$ be a vector in $[0,1]^6$. Then $\xi^6_{1,3,8}(x,\alpha)$ is the vector in $[0,1]^9$ that results from including in x the value α in positions 1, 3 and 8. i,e $\xi^n_{1,3,8}(x,\alpha) = (\alpha,x_1,\alpha,x_2,x_3,x_4,x_5,\alpha,x_6)$.

Once this notation is introduced, let us note that the equation associated to the notion of j-L strict stability can be reformulated as

$$A_{n+1}(\xi^n_j(x,\alpha)) = A_n(x) \;\; with\, \alpha = A_n(x) \;\; for\, x \in [0,1]^n.$$

Now it is possible to introduce the following definition:

Definition 3. Let $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ be a family of aggregation operators. Then, $\{A_n\}_n$ is a $r_1,\ldots,r_k$-L-strictly stable family for given values $r_1 < \ldots < r_k$ if

$$A_{n+k}\left(\xi^n_{r_1,r_2,\ldots,r_k}(x, A_n(x))\right) = A_n(x)$$

$\forall n \geqslant r_k - k$ and $\forall x \in [0,1]^n$

In a similar way as strict stability establishes some conditions between the members of a FAO when they differing one in cardinality (i.e. A_n and A_{n+1}), the previous definition establishes some relationships between the elements of a FAO when they differing k in cardinality (i.e. A_n and A_{n+k}), taking into account the way in which the information is aggregated (i.e. the structure of the data). In the following subsections, the consequences and links between strictly stable families of different orders are analyzed, distinguishing between the symmetric and non-symmetric case.

4.1. The symmetric case

In a similar way as it happens with j-L strict stability, when a *FAO* is symmetric (i.e. all its members are symmetric aggregation functions), the positions $r_1,\ldots,r_k$ are not relevant in the definition of order k strict stability. The only important parameter is the number k, that represents the order of stability being imposed. Therefore, in the symmetric case the definition of a strictly stable family of order k can be rewritten as

$$A_{n+k}\left(\xi^n_{n+1,\ldots,n+k}(x,A_n(x))\right)=A_n(x)$$

$\forall n \geqslant 2$ and $\forall x \in [0,1]^n$.

The following proposition analyzes the relationships between different orders of strictly stable families.

Proposition 1. *Let $\{A_n\}_n$ be a symmetric, strictly stable FAO. Then $\{A_n\}$ is a strictly stable family of order k for any $k \geqslant 2$.*

Proof. Let $\{A_n\}_n$ be a symmetric strictly stable FAO, and take $k \geqslant 2$. Also, let x be an element of $[0,1]^n$. Then, taking into account that $\{A_n\}_n$ is a strictly stable family, it follows that

$$A_n(x)=A_{n+1}(x,A_n(x))=A_{n+2}(x,A_n(x),A_{n+1}(x,A_n))$$

$$=A_{n+2}(x,A_n(x),A_n(x))=\ldots A_{n+k}(x,A_n(x),\ldots,A_n(x)),$$

concluding the proof.

Let us observe that the opposite is not necessarily true. Let $\{A_n\}_n$ be a FAO defined as

$$A_n(x_1,\ldots,x_n)=\begin{cases} Max\{x_1,\ldots,x_n\} & if\, n=2m \\ Min\{x_1,\ldots,x_n\} & if\, n=2m+1\end{cases}$$

It is easy to see that $\{A_n\}_n$ is a strictly stable family of order 2, but (as shown in [16]), $\{A_n\}_n$ is not a strictly stable family of order 1.

Corollary 1 The FAOs

- $\{Min_n(x_1,\ldots,x_n)=Min\{x_1,\ldots,x_n\}\forall n\in\mathbb{N}\}$,
- $\{Max_n(x_1,\ldots,x_n)=Max\{x_1,\ldots,x_n\}\forall n\in\mathbb{N}\}$,
- $\{Av_n(x_1,\ldots,x_n)=\sum_{i=1,n}\frac{x_i}{n}\forall n\in\mathbb{N}\}$,
- $\{G_n(x_1,\ldots,x_n)=(\prod_{i=1}^n x_i)^n\,\forall n\in\mathbb{N}\}$,
- $\{H_n(x_1,\ldots,x_n)=\frac{n}{\sum_{i=1,n}x_i}\forall n\in\mathbb{N}\}$

are strictly stable families of any order.

Proof. All these families are strictly stable FAOs (see [16] for more details), and then by Proposition 1 they are also strictly stable families of any order k.

Let us observe that a strictly stable family of order k is not in general a strictly stable family of order l, even when $k \leqslant l$. In the following proposition we establish a result that determines when this implication is true.

Proposition 2. *Let $\{A_n\}_n$ be a symmetric strictly stable FAO of order k, and let $l \leqslant k$. Then $\{A_n\}$ is a strictly stable family of order l if $l = mk$ for an integer m.*

Proof. Let $\{A_n\}_n$ be a symmetric strictly stable FAO of order k, let m be a positive integer and let $l = mk$. Given $x \in [0,1]^n$, we have to prove that:

$$A_{n+l}\left(x_1,\ldots,x_n,\overbrace{A_n(x),\ldots,A_n(x)}^{l\ times}\right)=A_n(x)$$

As $\{A_n\}_n$ is a symmetric strict stable family of order k, then the following holds:

$A_n(x) = A_{n+k}(x,\overbrace{A_n(x),\ldots,A_n(x)}^{k\ times})$, which is equal to

$$A_{n+2k}\left(\xi^{n+k}_{n+k,\ldots,n+2k}\left(\xi_{n+1,\ldots,n+k}(x,A_n(x)),A_{n+k}\left(\xi_{n+1,\ldots,n+k}(x,A_n(x))\right)\right)\right). \quad (1)$$

Now, as $A_n(x) = A_{n+k}\left(\xi_{n+1,\ldots,n+k}(x,A_n(x))\right)$, the expression given in (1) coincides with $A_{n+2k}(x,\overbrace{A_n(x),\ldots,A_n(x)}^{2k\ times})$. Therefore, at this point we already have that strict stability of order k implies strict stability of order $2k$. Following this sequence, we conclude that $A_n(x) = A_{n+mk}(x,\overbrace{A_n(x),\ldots,A_n(x)}^{mk\ times})$, and then the result is proved.

4.2. *Non symmetric case*

To conclude this section, now we present a result that establishes some connections between different orders of strict stability for non-symmetric FAOs.

Proposition 3. *Let $\{A_n\}_n$ be a j-L strictly stable FAO $\forall j \in \{r_1,\ldots,r_k\}$. Then $\{A_n\}$ is a $r_1,r_2,\ldots,r_k$ strictly stable family.*

Proof. Let $\{A_n\}_n$ be a j-L strictly stable FAO for all $j \in \{r_1,\ldots,r_k\}$. Let $x \in [0,1]^n$ be a n-dimensional vector, with $n > r_k - k$. As $\{A_n\}_n$ is $r_1 - L$ strictly stable family, then the following holds:

$$A_n(x)=A_{n+1}\left(\xi^n_{r_1}(x,A_n(x))\right).$$

Now, as $\{A_n\}_n$ is a r_2-L strict stable family, the previous expression is equal to

$$A_{n+2}\left(\xi_{r_2}^{n+1}\left(\xi_{r_1}^{n}(x,A_n(x)),A_{n+1}(\xi_{r_1}^{n}(x,A_n(x)))\right)\right),$$

which is equal to

$$A_{n+2}\left(\xi_{r_2}^{n+1}\left(\xi_{r_1}^{n}(x,A_n(x)),A_n(x)\right)\right),$$

which in turn coincides with

$$A_{n+2}\left(\xi_{r_1,r_2}^{n}(x,A_n(x))\right).$$

Following a similar reasoning, it can be proved iteratively that $A_n(x)$ coincides with $A_{n+k}\left(\xi_{r_1,\ldots,r_k}^{n}(x,A_n(x))\right)$, which concludes the proof.

Let us observe that the opposite is not necessarily true. For example, the family $\{A_n\}$ previously defined as the maximum for $n = 2m$ and the minimum for $n = 2m+1$ is a strictly stable family of order 2 for any positions $r_1 < r_2$ but is not a r_1 or r_2 strictly stable FAO.

5. Dealing with weights and missing data: an application of stability.

To illustrate an interesting application of the concept of stability let us introduce an example. Consider an undergraduate course for which the final mark is calculated as follows; 30% for assignments (3 assignments 10% each) and 70% final exam (so we have a weighted mean). Suppose that a student carried out all the tasks except the first assignment, and that he has reasonable grounds for exemption (he enrolled late into the subject). How his mark should be calculated in a fair and consistent manner with respect to the rest of the class? This constitutes a problem of not evaluable data, and for us a fair solution is to re-weigh the formula, so that the weights for the evaluable items are increased proportionally.

Let us recall again that our aim is not to decide how the vector of weights $w^4 = (w_1^4, w_2^4, w_3^4, w_4^4)$ should be, but to guarantee some stability or consistence in the aggregation process under different cardinalities. For example, it would seem rather inconsistent to choose $w^4 = (0.1, 0.1, 0.1, 0.7)$ when data regarding the four mentioned tests are available, but also choosing $w^3 = (0.8, 0.2, 0)$ in case the first test presents a missing value, since the relative importance of the tests is clearly different from one situation to another. From the point of view of consistency, this evaluation would not be stable.

We first focus our attention in the weighted mean aggregation family. This family, $\{W_n, n \in \mathbb{N}\}$, is defined through a vector of weights $w^n = (w_1^n, ..., w_n^n) \in [0,1]^n$ in such a way that $W_n(x_1, ..., x_n) = \sum_{i=1}^{n} w_i^n x_i$, where $\sum_{i=1}^{n} w_i^n = 1$ and $(x_1, ..., x_n) \in [0,1]^n \;\; \forall n$. In the weighted mean FAO, the weights associated to the elements being aggregated represent the *importance* of each one of these elements in the aggregation process. For this reason, the weighted mean surely is one of the most relevant and used aggregation operators in many different areas (e.g. statistics, knowledge representation problems, fuzzy logic, multiple criteria decision making, group decision making, etc.), and one of the most studied problems in all these areas is how to determine these *importances* or weights. The stability of this family was studied from a L-R point of view in [16]. Nevertheless, as we will see below, this study can not be directly applicable to the missing value problem in aggregation problems.

A missing data problem appears when for a specific object $x = (x_1, \ldots, x_n)$ one of its values x_i is missing. In the previous example it is $n = 4$, the information regarding an alternative is aggregated through $W_4(x_1, \ldots, x_4) = \sum_{i=1,4} w_i^4 x_i$, and the importance of the four criteria has been established by means of the four dimensional vector $w^4 = \left(w_1^4, w_2^4, w_3^4, w_4^4\right) = (0.1, 0.1, 0.1, 0.7)$. Now, consider an alternative x that presents the values $x = (\textit{not evaluable}, 0.3, 1, 1)$. What should be the aggregation operator A_3 to be used?

If we decide to use the weighted mean aggregation function for $n = 3$ (i.e. $A_3 = W_3$), the problem here is to determine the weights vector w^3. A possibility is to impose that W_3 and W_4 satisfy the strict stability conditions. Nevertheless, let us observe that the different strict stability conditions (L, R, $i-L$ or $j-R$ for different i and j positions) will give us different possibilities and solutions for the vector w^3. So, what stability constraint should we choose? Taking into account that the 1-th value x_1 is the one missing, it seems reasonable to impose the L (or equivalently the 4-R or 1-L) strict stability condition, i.e.

$$W_4(W_3(x_2,x_3,x_4),x_2,x_3,x_4) = W_3(x_2,x_3,x_4)$$

for any x_2, x_3 and x_4 in $[0,1]$

Concerning our example, this condition holds if and only if $w^3 = (\frac{1}{9},\frac{1}{9},\frac{7}{9})$. Let us observe that this vector maintains the relative proportions between the original weights for the non-missing values in the positions 2, 3 and 4.

In the previous example, the first value of the alternative $x = (\textit{not evaluable}, 0.3, 1, 1)$ is missing. But what should be the aggregation if the missing value is the second one? In general, and for non-symmetric FAOs where the position in which data appear is relevant, if there is some information $x = (x_1,\ldots,x_n)$ that has to be aggregated and we have a missing value x_j, we should impose strict j-L stability or equivalently $(n-(j+1))$-R strict stability to find the relations that should exist between the aggregation functions A_n and A_{n+1} in the whole family. Now, we present a proposition that gives sufficient conditions for the j-L-strict stability of the family $\{W_n\}_n$. Another proof of this proposition can be found in [2] and also in [9], in which similar conclusions are given when you have to solve missing data problems with the weighted mean family.

Proposition 4. *Let $w^n = (w_1^n,...,w_n^n) \in [0,1]^n, n \in \mathbb{N}$, be a sequence of weights of a weighted mean family $\{W_n\}_{n\in\mathbb{N}}$ such that $\sum_{i=1}^{n} w_i^n = 1$ holds $\forall n \geqslant 2$. Then, the family $\{W_n\}_{n\in\mathbb{N}}$ is a j-L-strictly stable family if and only if the sequence of weights satisfies*

$$\begin{cases} w_k^n = (1-w_j^n)\cdot(w_k^{n-1}) & for\ \ k = 1,\ldots,j-1 \\ w_{k+1}^n = (1-w_j^n)\cdot(w_k^{n-1}) & for\ \ k = j,\ldots,n-1 \end{cases}$$

$\forall n \in \mathbb{N}$.

Proof.

Note that for a generic weighted mean FAO $\{W_n\}_{n\in N}$ with weights w^n, $n \in \mathbb{N}$, the j-L-strict stability property can be restated as

$$\sum_{i=1}^{j-1}(w_i^n-(1-w_j^n)w_i^{n-1})x_i+\sum_{i=j}^{n-1}(w_{i+1}^n-(1-w_j^n)w_i^{n-1})x_i = 0$$

$\forall x_1,\ldots x_{n-1} \in [0,1]$.

From previous equation it is straightforward to conclude that the proposition holds.

5.1. *Dealing with more than one missing value*

To conclude the study of aggregation operators with missing values from the point of view of stability, we will try to extend the previous analysis to a situation in which more than one value could be missing. Let us suppose that we have two missing values in the positions $r < s$. That is, we have $x = (x_1,\ldots,x_{r-1},missing,x_{r+1},\ldots,x_{s-1},mising,x_{s+1},\ldots,x_n)$.

Following the equation of $r-s$-L strict stability, it is possible to build the aggregation operator A_{n-2} from A_n for a given n. Let us continue with the example of the undergraduate course evaluated through four tasks. Consider now a student who failed to hand in the second and third assignments because of a serious illness. If we decide to use the weighted mean aggregation function for $n = 2$ (i.e. $A_2 = W_2$), the problem is to determine the weights vector w^2 from w^4 (which is the available information). Then, it seems reasonable to impose the $2-3$-L stability condition to find the weights associated with the aggregation operator W_2 i.e:

$$W_4(x_1,W_2(x_1,x_2),W_2(x_1,x_2),x_2) = W_2(x_1,x_2)$$

for any x_1, x_2 in $[0,1]$.

For notational convenience, we have denoted by x_1 the value for the first variable and by x_2 the value for the fourth variable. So it is $x = (x_1,missing,missing,x_2)$. Then, the condition above holds if and only if $w^2 = (\frac{1}{8},\frac{7}{8})$. Let us observe that this vector maintains the relative proportions between the original weights for the non-missing values in the positions 1 and 4.

As done before for strict stability of order 1 with respect to the position j, now we analyze the conditions the family of weights should satisfy to guarantee strict stability of order 2 of the weighted mean FAO. This will enable us to state the relationships that should hold between weights of different dimensions to build a consistent aggregation process when we have missing data problems with more than one variable.

Proposition 5. *Let $w^n = (w_1^n, ..., w_n^n) \in [0,1]^n, n \in N$, be the sequence of weights of a weighted mean family $\{W_n\}_{n\in\mathbb{N}}$, verifying $\sum_{i=1}^{n} w_i^n = 1$ holds $\forall n \geqslant 2$. Then, the family $\{W_n\}_{n\in\mathbb{N}}$ is a $r_1 - r_2$-L-strictly stable family if and only if the sequence of weights satisfies*

$$\begin{cases} w_i^{n+2} = (1 - w_{r_1}^{n+2} - w_{r_2}^{n+2}) \cdot (w_i^n) & i = 1, \dots, r_1 - 1 \\ w_{i+1}^{n+2} = (1 - w_{r_1}^{n+2} - w_{r_2}^{n+2}) \cdot (w_i^n) & i = r_1, \dots, r_2 - 1 \\ w_{i+2}^{n+2} = (1 - w_{r_1}^{n+2} - w_{r_2}^{n+2}) \cdot (w_i^n) & i = r_2, \dots, n \end{cases}$$

$$\forall n \geqslant r_2 - 2 \in \mathbb{N}.$$

Proof.

Note that for a generic weighted mean FAO $\{W_n\}_{n\in\mathbb{N}}$ with weights w^n, $n \in \mathbb{N}$, the $r_1 - r_2$-L-strict stability property can be restated as

$$0 = |\sum_{i=1}^{r_1-1} x_i \left(w_i^{n+2} - (1 - w_{r_1}^{n+2} - w_{r_2}^{n+2}) \cdot (w_i^n)\right) + \dots$$

$$\dots \sum_{i=r_1 1}^{r_2-1} x_i \left(w_{i+1}^{n+2} - (1 - w_{r_1}^{n+2} - w_{r_2}^{n+2}) \cdot (w_i^n)\right) +$$

$$+ \sum_{i=r_2}^{n} x_i \left(w_{i+2}^{n+2} - (1 - w_{r_1}^{n+2} - w_{r_2}^{n+2}) \cdot (w_i^n)\right) |$$

.

From the previous equation it is straightforward to conclude that the proposition holds.

Proposition 6. *Let $w^n = (w_1^n, ..., w_n^n) \in [0,1]^n, n \in \mathbb{N}$, be a sequence of weights of a weighted mean family $\{W_n\}_{n\in\mathbb{N}}$ such that $\sum_{i=1}^{n} w_i^n = 1$ holds $\forall n \geqslant 2$. Then, the family $\{W_n\}_{n\in\mathbb{N}}$ is a $r_1, \dots, r_k$-L-strictly stable family if and only if the sequence of weights satisfies*

$$w_{i+f}^{n+k} = (1 - \sum_{v=1}^{k} w_{r_v}^{n+k}) \cdot (w_i^n),$$

for $f = 0, \dots, k$ and $i = r_f, \dots, r_{f+1} - 1$, where $r_0 = 1$ and $r_{k+1} = n + 1$ for notational convenience.

Proof. Similar to the proof of Proposition 6.

In order to extend the previous properties to a more general class of FAO, next we analyze strict stability of order k for transformations of the original FAO. However, let us first introduce the following notations and definitions.

Definition 4. Let $f : [0,1] \to A$ be a continuous and injective function, and let $\{\phi_n : A \to A,\ n \in \mathbb{N}\}$ be a family of aggregation operators defined in the domain A. Then, the transformed family of aggregation operators $\{M_f^{\phi_n}\}_{n\in\mathbb{N}}$ is defined as:

$$M_f^{\phi_n}(x_1, \dots, x_n) = f^{-1}\left(\phi_n\left(f(x_1), \dots, f(x_n)\right)\right)$$

Let us observe that if f is the identity function, then the transformed family coincides with the original family. If $\{\phi_n\}_{n\in\mathbb{N}}$ is the mean or the weighted mean then $M_f^{\phi_n}$ is called quasi-arithmetic mean or weighted quasi-arithmetic mean. The quasi-arithmetic mean functions are very important in many aggregation analysis. Some well-known quasi-arithmetic aggregation families are: the geometric mean (when $f(x) = log(x)$), the harmonic mean (when $f(x) = 1/x$) and the power mean (when $f(x) = x^p$), among others. It is important to remark that some of the most usual aggregation operators families (as for example the product $\{P_n\}_{n\in\mathbb{N}}$), can not be transformed or extended directly. For example if $f(x) = 5x$, then $A = [0,5]$, but it is not possible to guarantee that for all $n \in N$, $P_n\left(f(x_1), \dots, f(x_n)\right) = \prod_{i=1}^{n} f(x_i)$ belongs to the interval $[0,5]$.

In the next results, strict stability of different orders for transformations of the original FAO is analyzed. Particularly, it is shown that strict stability of order k remains after transformation.

Proposition 7. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{M_f^{\phi_n}\}_{n\in\mathbb{N}}$ be a family of aggregation operators and its extension or transformed aggregation. Then $\{M_f^{\phi_n}\}_{n\in N}$*

is a $r_1,\ldots,r_k j$-L-strictly stable family if and only if $\{\phi_n\}_{n\in\mathbb{N}}$ is a $r_1,\ldots,r_k j$-L-strictly stable family in the domain A.

Proof:

Taking into account that

$$M_f^{\phi_{n+k}}\left(\xi^n_{r_1,\ldots,r_k}(x, M_f^{\phi_n}(x))\right)$$

can be rewritten as

$$f^{-1}\left(\phi_{n+k}\left(\xi^n_{r_1,\ldots,r_k}(f(x),\phi_n(x))\right)\right),$$

the $r_1,\ldots,r_k j$-L strict stability condition for the FAO $\{M_f^{\phi_n}\}_{n\in\mathbb{N}}$ can be formulated as

$$f^{-1}\left(\phi_{n+k}\left(\xi^n_{r_1,\ldots,r_k}(f(x),\phi_n(x))\right)\right) = f^{-1}\left(\phi_n(f(x))\right).$$

Hence, since f is a continuous and injective function, such a condition holds if and only if $\{\phi_n\}_n$ is a strictly stable family in A, which concludes the proof.

Corollary 2 The weighted quasi-arithmetic mean FAO is a j-L-strictly stable family if and only if the sequence of weights satisfies $\forall n \in \mathbb{N}$

$$\begin{cases} w_k^n = (1-w_j^n)\cdot(w_k^{n-1}) & for\ \ k=1,\ldots,j-1 \\ w_{k+1}^n = (1-w_j^n)\cdot(w_k^{n-1}) & for\ \ k=j,\ldots,n-1 \end{cases}$$

Corollary 3 The weighted quasi-arithmetic mean FAO is a $r_1,\ldots,r_k j$-L-strictly stable family if and only if the sequence of weights satisfies

$$w_{i+f}^{n+k} = (1-\sum_{v=1}^{k} w_{r_v}^{n+k})\cdot(w_i^n),$$

for $f=0,\ldots,k$ and $i=r_f,\ldots,r_{f+1}-1$, where $r_0=1$ and $r_{k+1}=n+1$ for notational convenience.

6. Final Comments

In this paper we have developed previous works of the authors concerning the key issue of the relationships that should hold between the operators in a family of aggregation operators in order to understand they properly define a *consistent* whole. Particularly, we have extended the basic notions of L and R strict stability of a family of aggregation operators presented in [16,11,15] to the more general framework of i-L and j-R strict stability. Also, we have introduced the notion of strict stability of order k, that constitutes a further extension of the previous notions as it enables to relate operators of arbitrarily different cardinalities. Moreover, all these notions have been analyzed more deeply in relation with the weighted mean and weighted quasi-arithmetic means families. In addition, we present an interesting application of these strict stability notions and their related constraints or conditions in order to deal with missing data problems in an aggregation operator framework.

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