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On (M,N)-SI (implicative) filters in R_0-algebras

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Abstract

Molodtsov's soft set theory provides a general mathematical framework for dealing with uncertainty. This paper aims to put forward a new soft set–(M,N)-soft intersection set, which is a generalization of soft intersection sets. We introduce (M,N)-SI (implicative) filters of R_0-algebras. Some characterizations of these kinds of filters are established. In particular, we discuss the properties of (M,N)-soft congruences in R_0-algebras. It can lay a foundation for providing a new soft algebraic tool in considering many problems that contain uncertainties.

Keywords: Soft set; R_0-algebras; filter; implicative filter; (M,N)-soft congruence; (M,N)-SI implicative(Boolean) filter.

1. Introduction

The concept of R_0-algebras was first introduced by Wang[39] by providing an algebraic proof of the completeness theorem of a formal deductive system[37,38,31,45]. In 2005, Liu and Li[21,22] have extended the notions of implicative filters and Boolean filters to R_0-algebras by considering the fuzzification of such notions. It can be easily observed that R_0-algebras are different from the BL-algebras[14,46] because the identity $x \wedge y = x \odot (x \to y)$ holds in BL-algebras, but it does not hold in R_0-algebras. We note that R_0-algebras are also different from the lattice implication algebras[41,42] because the identity $(x \to y) \to y = (y \to x) \to x$ holds in lattice implication algebras, but it does not hold in R_0-algebras. Although they are essentially different, they still have some similarities, that is, they all have the implication operator $\to$. Therefore, it is meaningful to generalize the lattice implication algebras and BL-algebras to R_0-algebras. In [7], Esteva and Godo introduced the MTL-algebra, which is an algebra in-

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duced by using a continuous t-norm and its corresponding residuum. It can be proved that an R_0-algebra is, in particular a MTL-algebra in which its t-norm $\odot$ is a nilpotent minimum t-norm [7]. In particular, Ma[25,27] discussed fuzzy filters of R_0-algebras.

It is well known that the complexities of modelling uncertain data in economics, engineering, environmental science, sociology, information sciences and many other fields can not be successfully dealt with by classical methods. Although probability theory, fuzzy set theory and rough set theory are well-known and effective tools for dealing with vagueness and uncertainty. Each of them has certain inherent limitations. Based on this reason, Molodtsov [30] proposed a completely new approach for modeling vagueness and uncertainty, which is called soft set theory. Since then, especially soft set operations, have undergone tremendous studied, such as [2,3,10,11,28,32,33,34].

We note that soft set theory emphasizes a balanced coverage of both theory and practice. Nowadays, it has promoted a breath of the discipline of information sciences, intelligent systems, expert and decision support systems, expert and decision support systems, knowledge systems and decision making, and so on. For examples, see[5,6,8,12,13,15,16,17,29,40,47]. At the same time, soft set theory has been found its wide-ranging applications in the algebraic structures, such as [1,9,18,20,23,24,26,43,44]. Recently, Çağman and Sezgin [4,35] made a new approach to soft intersection theory to groups and near-rings. Further, Jun et al.[19] applied this idea to R_0-algebras. They introduced the concept of soft intersection filters of R_0-algebras. Some new characterizations were provided.

The present paper is organized as follows. In section 2, we recall some concepts and results of R_0-algebras and soft sets. In section 3, we investigate some characterizations of (M,N)-SI filters of R_0-algebras. In particular, some important properties of (M,N)-soft congruences of R_0-algebras are discussed in section 4. Finally, we study (M,N)-SI implicature(Boolean) fillers of R_0-algebras. It is shown that (M,N)-SI Boolean filters and (M,N)-SI implicature fillers of R_0-algebras are equivalent in section 5.

2. Preliminaries

By an R_0-algebra[39], we mean a bounded lattice $L = (L, \leqslant, \wedge, \vee, ', \rightarrow, 0, 1)$, which $'$ is an order-reversing involution and with a binary operation $\rightarrow$ such that the following conditions hold:

(R_1) $x \rightarrow y = y' \rightarrow x'$;
(R_2) $1 \rightarrow x = x$;
(R_3) $(y \rightarrow z) \wedge ((x \rightarrow y) \rightarrow (x \rightarrow z)) = y - x$;
(R_4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
(R_5) $x \rightarrow (y \vee z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$;
(R_6) $(x \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow (x' \vee y)) = 1$.

In any R_0-algebra L, the following statements are true(see [31]):

(a_1) $x \leqslant y \Leftrightarrow x \rightarrow y = 1$.
(a_2) $x \leqslant y \rightarrow x$.
(a_3) $x' = x \rightarrow 0$.
(a_4) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.
(a_5) $x \leqslant y \Rightarrow x \rightarrow z \geqslant y \rightarrow z$.
(a_6) $x \leqslant y \Rightarrow z \rightarrow x \leqslant z \rightarrow y$.
(a_7) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.
(a_8) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$.
(a_9) $x \odot x' = 0, x \oplus x' = 1$.
(a_{10}) $x \odot y \leqslant x \wedge y, x \odot (x \rightarrow y) \leqslant x \wedge y$.
(a_{11}) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$.
(a_{12}) $x \leqslant y \rightarrow (x \odot y)$.
(a_{13}) $x \odot y \leqslant z \Leftrightarrow x \leqslant y \rightarrow z$.
(a_{14}) $x \leqslant y \Rightarrow x \odot z \leqslant y \odot z$.
(a_{15}) $x \rightarrow y \leqslant (y \rightarrow z) \rightarrow (x \rightarrow z)$.
(a_{16}) $(x \rightarrow y) \odot (y \rightarrow z) \leqslant x \rightarrow z$.

Let L be an R_0-algebra. For any $x, y \in L$, define $x \odot y = (x \rightarrow y')'$ and $x \oplus y = x' \rightarrow y$. It is proved that $\odot$ and $\oplus$ are commutative, associative and $x \oplus y = (x' \odot y')'$, and $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice.

Now, we recall some basic concepts of filters in R_0-algebras.

A non-empty subset F of L is called a filter of L if it satisfies (F_1) $1 \in F$ and (F_2) $x, x \rightarrow y \in F \Rightarrow y \in F$. A filter F of L is called a Boolean filter of L if $(F_3) x \vee x' \in F$, for all $x \in L$. A non-empty subset F of L is called an implicative filter of L if it satisfies (F_1) and (F_4) $x \rightarrow (y \rightarrow z) \in F, x \rightarrow y \in A \Rightarrow x \rightarrow z \in F$. We know a filter of L is Boolean if and only it is implicative(see [31,21,22].)

From now on, L is an R_0-algebra, U is an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A,B,C \subseteq E$.

Definition 1. [30,5] A soft set f_A over U is a set defined by $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs $f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\}$. It is clear to see that a soft set is a parameterized family of subsets of U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2. [5] Let $f_A, f_B \in S(U)$. Then,

(1) f_A is called soft subset of f_B and denoted by $f_A \widetilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$, for all $x \in E$. f_A and f_B are called soft equal, denoted by $f_A = f_B$, if $f_A \widetilde{\subseteq} f_B$ and $f_A \widetilde{\supseteq} f_B$;

(2) The union of f_A and f_B, denoted by $f_A \widetilde{\cup} f_B$, is defined as $f_A \widetilde{\cup} f_B = f_{A\cup B}$, where $f_{A\cup B}(x) = f_A(x) \cup f_B(x)$, for all $x \in E$;

(3) The intersection of f_A and f_B, denoted by $f_A \widetilde{\cap} f_B$, is defined as $f_A \widetilde{\cap} f_B = f_{A\cap B}$, where $f_{A\cap B}(x) = f_A(x) \cap f_B(x)$, for all $x \in E$.

Definition 3. [19] (1) A soft set f_L over U is called an SI-filter of L over U if it satisfies:

(S_1) $f_L(x) \subseteq f_L(1)$ for any $x \in L$.

(S_2) $f_L(x \to y) \cap f_L(x) \subseteq f_L(y)$ for all $x, y \in L$.

(2) A soft set f_L over U is called an SI-implicative filter of L over U if it satisfies (S_1) and

(S_3) $f_L(x \to (y \to z)) \cap f_L(x \to y) \subseteq f_L(x \to z)$ for all $x, y, z \in L$.

Remark 1. In [19], Jun et al. called these two concepts int-soft filters and int-soft implicative filters, respectively. But it was first introduced this concept by Çağman [4]. In the following paper, we will use the terminology in [4].

3. (M,N)-SI filters

In this section, we introduce the concept of (M,N)-SI filters of R_0-algebras and investigate some characterizations. From now on, $\emptyset \subseteq M \subset N \subseteq U$.

Definition 4. A soft set f_S over U is called an (M,N)-soft intersection filter (briefly, (M,N)-SI filter) of L over U if it satisfies:

(SI_1) $f_L(x) \cap N \subseteq f_L(1) \cup M$ for all $x \in L$;

(SI_2) $f_L(x \to y) \cap f_L(x) \cap N \subseteq f_L(y) \cup M$ for all $x, y \in L$

Remark 2. If f_L is an SI-filter of L over U, then f_L is an $(\emptyset, U)$-SI filter of L over U. Hence, every SI-filter of L is an (M,N)-SI filter of L, but the converse is not true.

Example 1. Assume that $U = S_3$, symmetric group, is the universal set and let $L = \{0, a, b, c, 1\}$, where $0 < a < b < c < 1$. Define $'$ and $\to$ as follows:

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

Then $(L, \wedge, \vee, ', \to)$ is an R_0-algebra[21], where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

Let $M = \{(13), (123)\}$ and $N = \{(1), (12), (13), (123)\}$.

Define a soft set f_L over U by

$f_L(1) = \{(1), (12), (123)\}$, $f_L(c) = \{(1), (12), (13), (123)\}$ and $f_L(a) = f_L(b) = f_L(0) = \{(1), (12)\}$.

Then one can easily check that f_L is an (M,N)-SI filter of L over U, but it is not SI-filter of L over U since $f_L(c) \nsubseteq f_L(1)$.

The following proposition is obvious.

Proposition 1. *If a soft set f_L over U is an (M,N)-SI filter of L over U, then*

$(f_S(1) \cap N) \cup M \supseteq (f_S(x) \cap N) \cup M$ *for all* $x \in S$.

Proposition 2. *If f_L is an (M,N)-SI filter of L over U, then $f_L^* = \{x \in L | (f_L(x) \cap N) \cup M = (f_L(1) \cap N) \cup M\}$ is a filter of L.*

Proof. *Assume that f_L is an (M,N)-SI filter of L over U, then it is clear that $1 \in f_L^*$.*

For any $x, x \to y \in f_L^*$, then

$(f_L(x) \cap N) \cup M = (f_L(x \to y) \cap N) \cup M = (f_L(1) \cap N) \cup M$.

By Proposition 2, we have $(f_L(y) \cap N) \cup M \subseteq (f_L(1) \cap N) \cup M$.

Since f_L is an (M,N)-SI filter of L over U, we have

$$\begin{aligned}(f_L(y)\cap N)\cup M &= ((f_L(y)\cup M)\cap N)\cup M\\ &\supseteq (f_L(x)\cap f_L(x\to y)\cap N)\cup M\\ &= ((f_L(y)\cap N)\cup M)\cap((f_L(x\to y)\cap N)\cup M)\\ &= (f_L(1)\cap N)\cup M.\end{aligned}$$

Hence, $(f_L(y)\cap N)\cup M = (f_L(1)\cap N)\cup M$, *which implies,* $y\in f_L^*$. *This implies that* f_L^* *is a filter of L.*

Define an ordered relation "$\widetilde{\subseteq}_{(M,N)}$" *on* $S(U)$ *as follows:*

For any $f_L, g_L \in S(U), \emptyset\subseteq M\subset N\subseteq U$, *we define* $f_L\widetilde{\subseteq}_{(M,N)}g_L \Leftrightarrow f_L\cap N\widetilde{\subseteq} g_S\cup M$.

And we define a relation " $=_{(M,N)}$ " *as follows:* $f_L =_{(M,N)} g_L \Leftrightarrow f_L\widetilde{\subseteq}_{(M,N)} g_L$ *and* $g_L\widetilde{\subseteq}_{(M,N)} f_L$. □

Then, we can denote Definition 4 as follows:

Definition 5. A soft set f_L over U is called an (M,N)-soft intersection filter(briefly, (M,N)-SI filter) of L over U if it satisfies:

(SI_1') $f_L(x)\widetilde{\subseteq}_{(M,N)}f_L(1)$ for all $x\in L$;

(SI_2') $f_L(x\to y)\cap f_L(x)\widetilde{\subseteq}_{(M,N)}f_L(y)$ for all $x,y\in L$.

Proposition 3. *If a soft set* f_L *over U is an* (M,N)-SI*-filter of L, then for any* $x,y,z\in L$.

(1) $x\leqslant y\Rightarrow f_L(x)\widetilde{\subseteq}_{(M,N)}f_L(y)$.

(2) $f_L(x\to y) = f_L(1)\Rightarrow f_L(x)\widetilde{\subseteq}_{(M,N)}f_L(y)$.

(3) $f_L(x\odot y) =_{(M,N)} f_L(x)\cap f_L(y) =_{(M,N)} f_L(x\wedge y)$.

(4) $f_L(0) =_{(M,N)} f_L(x)\cap f_L(x')$.

(5) $f_L(x\to y)\cap f_L(y\to z)\widetilde{\subseteq}_{(M,N)}f_L(x\to z)$.

(6) $x\odot y\leqslant z\Rightarrow f_L(x)\cap f_L(y)\widetilde{\subseteq}_{(M,N)}f_L(z)$.

(7) $f_L(x\to(z'\to y))\cap f_L(y\to z)\widetilde{\subseteq}_{(M,N)}f_L(x\to(z'\to z))$.

(8) $f_L(x\to(y\to z))\cap f_L(x\to y)\widetilde{\subseteq}_{(M,N)}f_L(x\to(x\to z))$.

Proof. (1) *Let* $x,y\in L$ *be such that* $x\leqslant y$, *then* $x\to y = 1$, *and so*

$$\begin{aligned}(f_L(x)\cap N) &= (f_L(x)\cap N)\cap(f_L(1)\cup M)\\ &= (f_L(y)\cap N)\cap(f_L(x\to y)\cup M)\\ &\subseteq (f_L(x)\cap f_L(x\to y)\cap N)\cup M\\ &\subseteq f_L(y)\cup M,\end{aligned}$$

which implies, $f_L(x)\widetilde{\subseteq}_{(M,N)}f_L(y)$.

(2) *Let* $x,y\in L$ *be such that* $f_L(x\to y) = f_L(1)$, *then*

$$\begin{aligned}f_L(x)\cap N &= (f_L(x)\cap N)\cap(f_L(1)\cup M)\\ &= (f_L(x)\cap N)\cap(f_L(x\to y)\cup M)\\ &\subseteq (f_L(x)\cap f_L(x\to y)\cap N)\cup M\\ &\subseteq f_L(y)\cup M,\end{aligned}$$

that is, $f_L(x)\widetilde{\subseteq}_{(M,N)}f_L(y)$.

(3) *By* (a_9), $x\odot y\leqslant x\wedge y$ *for all* $x,y\in L$, *then by* (1), $f_L(x\odot y)\widetilde{\subseteq}_{(M,N)}f_L(x)\cap f_L(y)$. *On the other hand, by* (a_{11}) *and* (1), $f_L(x)\widetilde{\subseteq}_{(M,N)}f_L(y\to(x\odot y))$. *It follows from* (SI_2) *that* $f_L(x)\cap f_L(y)\widetilde{\subseteq}_{(M,N)}f_L(y\to(x\odot y))\cap f_L(y)\subseteq f_L(x\odot y)$. *Hence* $f_L(x\odot y) =_{(M,N)} f_L(x)\cap f_L(y)$.

By (a_2) *and* (a_9), *we have* $y\leqslant x\to y$ *and* $x\odot(x\to y)\leqslant x\wedge y$. *Then* $f_L(y)\widetilde{\subseteq}_{(M,N)}f_L(x\to y)$ *and* $f_L(x\odot(x\to y))\widetilde{\subseteq}_{(M,N)}f_L(x\wedge y)$. *Hence, we have*

$f_L(x)\cap f_L(y)\widetilde{\subseteq}_{(M,N)}f_L(x)\cap f_L(x\to y) =_{(M,N)} f_L(x\odot(x\to y))\widetilde{\subseteq}_{(M,N)}f_L(x\wedge y)\widetilde{\subseteq}_{(M,N)}f_L(x)\cap f_L(y)$,

which implies, $f_L(x)\cap f_L(y) =_{(M,N)} f_L(x\wedge y)$.

Thus, $f_L(x\odot y) =_{(M,N)} f_L(x)\cap f_L(y) =_{(M,N)} f_L(x\wedge y)$.

(4) *Since* $x\odot x' = 0$, *then it is a consequence of* (3).

(5) *By* (6) *and* (a_{15}), (1) *and* (3), *we can deduce it.*

(7) *By* (a_{10}) *and* (a_{15}), *we have* $(x\to(z'\to y))\odot(y\to z) = ((x\odot z')\to y)\odot(y\to z)\leqslant(x\odot z')\to z = x\to(z'\to z)$. *By* (1) *and* (3), *we deduce that* $f_L(x\to(z'\to y))\cap f_L(y\to z) =_{(M,N)} f_L(x\to(z'\to y))\odot(y\to z)\widetilde{\subseteq}_{(M,N)}f_L(x\to(z'\to z))$.

(8) *By* (R_4) *and* (a_{12}), *we have*

$(x\to(y\to z))\odot(x\to y) = (y\to(x\to z))\odot(x\to y)\leqslant x\to(x\to z)$.

Hence, by (1) *and* (3), *we can deduce that*

$f_L(x\to(y\to z))\cap f_L(x\to y) =_{(M,N)} f_L(x\to(y\to z))\odot(x\to y)\widetilde{\subseteq}_{(M,N)}f_L(x\to(x\to z))$. □

Theorem 4. A soft set f_L *over U is an* (M,N)-SI *filter of L over U if and only if it satisfies:*

(SI_3) $\forall x,y\in L$, $x\leqslant y\Rightarrow f_L(x)\widetilde{\subseteq}_{(M,N)}f_L(y)$.

(SI_4) $\forall x,y\in L, f_L(x\odot y) =_{(M,N)} f_L(x)\cap f_L(y)$.

Proof. " $\Rightarrow$ " *By Proposition 3*(1) *and* (3).

" $\Leftarrow$ " *Let* $x,y\in L$. *Since* $x\leqslant 1$, *then by* (SI_3), *we have* $f_L(x)\widetilde{\subseteq}_{(M,N)}f_L(1)$, *that is* , $f_L(x)\cap N\subseteq f_L(1)\cup$

M. This implies that (SI_1) holds.

By (a_9), $x \odot (x \to y) \leqslant y$. Hence, by (SI_3) and (SI_4), $f_L(x) \cap f_L(x \to y) =_{(M,N)} f_L(x \odot (x \to y)) \widetilde{\subseteq}_{(M,N)} f_L(y)$, that is, $f_L(x) \cap f_L(x \to y) \cap N \subseteq f_L(y) \cup M$. This implies that (SI_2) holds.

Therefore, f_L is an (M,N)-SI filter of L over U. □

The following proposition is obvious.

Proposition 5. *A soft set f_L over U is an (M,N)-SI filter of L over U if and only if satisfies:*
(SI_5) $x \leqslant y \to z \Rightarrow f_L(x) \cap f_L(y) \widetilde{\subseteq}_{(M,N)} f_L(z)$.

Theorem 6. *If a soft set f_L over U is an (M,N)-SI filter of L over U, then the following are equivalent:*
(i) $\forall x,y,z \in L$, $f_L(x \to (y \to z)) \cap f_L(x \to y) \widetilde{\subseteq}_{(M,N)} f_L(x \to z)$.
(ii) $\forall x,y \in L$, $f_L(x \to (x \to y)) \widetilde{\subseteq}_{(M,N)} f_L(x \to y)$.
(iii) $\forall x,y,z \in L$, $f_L(x \to (y \to z)) \widetilde{\subseteq}_{(M,N)} f_L((x \to y) \to (x \to z))$.

Proof. *(i) $\Rightarrow$ (ii) Putting $z = y$ and $z = x$ in (1) and using (SI_1), we can deduce that*

$$\begin{aligned} f_L(x \to y) \cup M &= (f_L(x \to y) \cup M) \cup M \\ &\supseteq (f_L(x \to (x \to y)) \cap f_L(x \to x) \cap N) \cup M \\ &= (f_L(x \to (x \to y)) \cap f_L(1) \cap N) \cup M \\ &\supseteq f_L(x \to (x \to y)) \cap (f_L(1) \cup M) \cap N \\ &\supseteq f_L(x \to (x \to y)) \cap N \end{aligned}$$

that is, $f_L(x \to (x \to y)) \widetilde{\subseteq}_{(M,N)} f_L(x \to y)$.
(ii) $\Rightarrow$ (iii) By (a_5) and (a_{15}), we have
$x \to (y \to z) \leqslant x \to ((x \to y) \to (x \to z))$, and so
$f_L(x \to (y \to z)) \cap N \subseteq f_L(x \to ((x \to y) \to (x \to z))) \cup M$.
Then

$$\begin{aligned} & f_L((x \to y) \to (x \to z)) \cup M \\ &= f_L(x \to ((x \to y) \to z)) \cup M \\ &= (f_L(x \to ((x \to y) \to z)) \cup M) \cup M \\ &\supseteq (f_L(x \to (x \to ((x \to y) \to z))) \cap N) \cup M \\ &= (f_L(x \to ((x \to y) \to (x \to z))) \cap N) \cup M \\ &\supseteq (f_L(x \to ((x \to y) \to (x \to z))) \cup M) \cap N \\ &\supseteq (f_L(x \to (x \to z)) \cap N, \end{aligned}$$

that is, $f_L(x \to (y \to z)) \widetilde{\subseteq}_{(M,N)} f_L((x \to y) \to (x \to z))$.

(iii) $\Rightarrow$ (i) By (SI_2) and (iii), we have

$$\begin{aligned} & f_L(x \to z) \cup M \\ &= (f_L(x \to z)) \cup M) \cup M \\ &\supseteq (f_L((x \to y) \to (x \to z)) \cap f_L(x \to y) \cap N) \cup M \\ &\supseteq (f_L((x \to y) \to (x \to z)) \cup M) \cap N \\ &\supseteq f_L(x \to (y \to z)) \cap f_L(x \to y) \cap N. \end{aligned}$$

that is, $f_L(x \to (y \to z)) \cap f_L(x \to y) \widetilde{\subseteq}_{(M,N)} f_L(x \to z)$. □

4. (M,N)-soft congruences

In this section, we investigate (M,N)-soft congruences, (M,N)-soft congruences classes and quotient soft R_0-algebras.

Definition 6. A soft relation θ from $f_L \times f_L$ to $P(U \times U)$ is called an (M,N)-congruence in L over $U \times U$ if it satisfies:
(C_1) $\theta(1,1) =_{(M,N)} \theta(x,x)$, $\forall\, x \in L$.
(C_2) $\theta(x,y) =_{(M,N)} \theta(y,x)$, $\forall\, x \in L$.
(C_3) $\theta(x,y) \cap \theta(y,z) \widetilde{\subseteq}_{(M,N)} \theta(x,z)$, $\forall\, x,y,z \in L$.
(C_4) $\theta(x,y) \widetilde{\subseteq}_{(M,N)} \theta(x \odot z, y \odot z)$, $\forall\, x,y,z \in L$.
(C_5) $\theta(x,y) \widetilde{\subseteq}_{(M,N)} \theta(x \to z, y \to z) \cap \theta(z \to x, z \to y)$, $\forall\, x,y,z \in L$.

Definition 7. Let θ be an (M,N)-congruence in BL-algebra L over $U \times U$ and $x \in L$. Define θ^x in L as $\theta^x(y) = \theta(x,y)$, $\forall y \in L$. The set θ^x is called an (M,N)-congruence class of x by θ in L. The set $L/\theta = \{\theta^x | x \in L\}$ is called a quotient soft set by θ.

Lemma 7. *If θ is an (M,N)-congruence in L over $U \times U$, then $\theta(x,y) \widetilde{\subseteq}_{(M,N)} \theta(1,1)$, $\forall x,y \in L$.*
Proof. *By (C_1) and (C_3), we have $\theta(1,1) = \theta(x,x) \widetilde{\supseteq}_{(M,N)} \theta(x,y) \cap \theta(y,x) = \theta(x,y)$.* □

Lemma 8. *If θ is an (M,N)-congruence in L over $U \times U$, then θ^1 is an (M,N)-SI filter of L over U.*
Proof. *For any $x \in L$, we have*
$\theta^1(1) = \theta(1,1) \widetilde{\supseteq}_{(M,N)} \theta(1,x) = \theta^1(x)$.
This proves that (SI'_1) holds.
For any $x,y \in L$, by (C_3) and (C_5), we have $\theta(1,y) \widetilde{\supseteq}_{(M,N)} \theta(1,x \to y) \cap \theta(x \to y,y)$ and $\theta(x \to y,y) = \theta(x \to y, 1 \to y) \widetilde{\supseteq}_{(M,N)} \theta(x,1)$.
Then

$\theta(1,y)\widetilde{\supseteq}_{(M,N)}\theta(1,x \to y) \cap \theta(x,1) = \theta(1,x) \cap \theta(1,x \to y)$,

this is, $\theta^1(y)\widetilde{\supseteq}_{(M,N)}\theta^1(x)\cap\theta^1(x \to y)$. *This proves that* (SI_2') *holds. Thus,* θ^1 *is an* (M,N)-SI *filter of* L *over* U. □

Lemma 9. *Let* f_L *be an* (M,N)-SI *filter of* L *over* U, *then* $\theta(x,y)=f_L(x \to y)\cap f_L(y \to x)$ *is an* (M,N)-*soft congruence in* L.

Proof. *For any* $x,y,z \in L$, *we have*

(1) $\theta_f(1,1) = f_L(1 \to 1)\cap f_L(1 \to 1) = f_L(1) = f_L(x \to x)\cap f_L(x \to x) = \theta_f(x,x)$. *This proves that* (C_1) *holds.*

(2) *It is clear that* (C_2) *holds.*

(3) *By Proposition 3(5), we have*

$$\begin{aligned}&\theta_f(x,y)\cap\theta_f(y,z)\\ &= (f_L(x \to y)\cap f_L(y \to x))\cap(f_L(y \to z)\cap f_L(z \to y))\\ &= (f_L(x \to y)\cap f_L(y \to z))\cap(f_L(y \to x)\cap f_L(z \to y))\\ &= f(N_1(\widetilde{A}^+,\widetilde{B}^+)),N_1(\widetilde{A}^-,\widetilde{B}^-))\\ &\widetilde{\subseteq}_{(M,N)}f_L(x \to z)\cap f_L(z \to x)\\ &= \theta_f(x,z).\end{aligned}$$

Thus, (C_3) *holds.*

(4) *Since* $x \to y \leqslant (x\odot z) \to (y\odot z)$ *and* $y \to x \leqslant (y\odot z) \to (x\odot z)$, *we have* $f_L(x \to y)\widetilde{\subseteq}_{(M,N)}f_L((x\odot z) \to (y\odot z))$ *and* $f_L(y \to z)\widetilde{\subseteq}_{(M,N)}f_L((y\odot z) \to (x\odot z))$.

Thus, $f_L(x \to y)\cap f_L(y \to x)\widetilde{\subseteq}_{(M,N)}f_L((x\odot z) \to (y\odot z))\cap f_L((y\odot z) \to (x\odot z))$.

which implies, $\theta_f(x,y)\widetilde{\subseteq}_{(M,N)}\theta_f(x\odot z,y\odot z)$. *This implies that* (C_4) *holds.*

(5)
$$\begin{aligned}&\theta_f(x \to z,y \to z)\cap\theta_f(z \to x,z \to y)\\ &= f_L((x \to z) \to (y \to z))\cap f_L((y \to z) \to (x \to z))\\ &\cap f_L((z \to x) \to (z \to y)\cap f_L((z \to y) \to (z \to x))\\ &\widetilde{\supseteq}_{(M,N)}f_L(y \to x)\cap f_L(x \to y)\\ &= \theta_f(x,y).\end{aligned}$$

Thus, (C_5) *holds. Therefore,* θ_f *is an* (M,N)-*soft congruence in* L. □

Let f_L be an (M,N)-SI filter of L over U and $x \in L$. In the following, let f^x denote the (M,N)-congruence class of x by θ_f in L and L/f the quotient soft set by θ_f.

Lemma 10. *If* f_L *is an* (M,N)-SI *filter of* L *over* U, *then* $f^x =_{(M,N)} f^y$ *if and only if* $f_L(x \to y) =_{(M,N)} f_L(y \to x) =_{(M,N)} f_L(1)$ *for all* $x,y \in L$.

Proof. *If* f_L *is an* (M,N)-SI *filter of* L *over* U, *then* $f^\mu(\nu) = \theta_f^\mu(\nu) = \theta_f(\mu,\nu) = f_L(\mu \to \nu)\cap f_L(\nu \to \mu)$, *that is,* $f^\mu(\nu) = f_L(\mu \to \nu)\cap f_L(\nu \to \mu)$ *for all* $x,y \in L$. *If* $f^x =_{(M,N)} f^y$, *then* $f^x(x) =_{(M,N)} f^y(x)$, *and so,* $f_L(x \to x) = f_L(1) =_{(M,N)} f_L(y \to x)\cap f_L(x \to y)$. *Thus,* $f_L(y \to x) =_{(M,N)} f_L(x \to y) =_{(M,N)} f_L(1)$.

Conversely, assume the given condition holds. By Proposition 3, we have

$f_L(x \to z)\widetilde{\supseteq}_{(M,N)}f_L(x \to y)\cap f_L(y \to z)$ *and* $f_L(y \to z)\widetilde{\supseteq}_{(M,N)}f_L(y \to x)\cap f_L(x \to z)$.

If $f_L(y \to x) =_{(M,N)} f_L(x \to y) =_{(M,N)} f_L(1)$, *then* $f_L(x \to z) \supseteq_{(M,N)} f_L(y \to z)$ *and* $f_L(y \to z) \supseteq_{(M,N)} f_L(x \to z)$. *Thus,* $f_L(x \to z) =_{(M,N)} f_L(y \to z)$. *Similarly, we can prove that* $f_L(z \to x) =_{(M,N)} f_L(z \to y)$. *This implies that*

$f^x(z) = f_L(x \to z)\cap f_L(z \to x) =_{(M,N)} f_L(y \to z)\cap f_L(z \to y) = f_L^y(z)$ *for all* $z \in L$.

Hence $f^x =_{(M,N)} f^y$. □

Denote $f_{f(1)} = \{x \in L|f(x) =_{(M,N)} f(1)\}$.

Corollary 11. *If* f *is an* (M,N)-SI *filter of* L *over* U, *then* $f^x =_{(M,N)} f^y$ *if and only if* $x \sim_{f_{f(1)}} y$, *where* $x \sim_{f_{f(1)}} y$ *if and only if* $x \to y \in f_{f(1)}$ *and* $y \to x \in f_{f(1)}$.

Let f be an (M,N)-SI filter of L over U. For any $f^x, f^y \in L/f$, we define

$$f^x \vee f^y =_{(M,N)} f^{x\vee y},\ \ f^x \wedge f^y =_{(M,N)} f^{x\wedge y},$$
$$(f^x)' =_{(M,N)} f^{x'} \text{ and } f^x \to f^y =_{(M,N)} f^{x\to y}.$$

Theorem 12. *Let* f *be an* (M,N)-SI *filter of* L *over* U, *then* $L/f = (L/f,\wedge,\vee,',\to,f^0,f^1)$ *is an* R_0-*algebra.*

Proof. *We can claim that the above operations on* L/f *are well-defined. In fact, if* $f^x =_{(M,N)} f^y$ *and* $f^a =_{(M,N)} f^b$, *then by Corollary 11, we have* $x \sim_{f_{f(1)})} y$ *and* $a \sim_{f_{f(1)}} b$, *and so* $x \vee a \sim_{f_{f(1)}} y \vee b$. *Thus,* $f^{x\vee a} =_{(M,N)} f^{y\vee b}$. *Similarly, we can prove* $f^{x\wedge a} =_{(M,N)} f^{y\wedge b}, (f^x)' = f^{x'}$ *and* $f^{x\to a} =_{(M,N)} f^{y\to b}$. *Then we can easily check that* L/f *is an* R_0-*algebra.* □

Theorem 13. *Let* f_L *be an* (M,N)-SI *filter of* L *over* U, *then* $L/f \cong L/f_{f(1)}$.

Proof. *Define* $\varphi : L \to L/f$ *by* $\varphi(x) = f^x$ *for all* $x \in L$.

For any $x,y \in L, \varphi(x \vee y) = f^{x\vee y} =_{(M,N)} f^x \vee f^y = \varphi(x) \vee \varphi(y), \varphi(x \wedge y) = f^{x\wedge y} =_{(M,N)} f^x \wedge f^y = \varphi(x) \wedge \varphi(y), \varphi(x') = f^{x'} =_{(M,N)} (f^x)' = (\varphi(x))'$ *and* $\varphi(x \to$

$y) = f^{x\to y} =_{(M,N)} f^x \to f^y = \varphi(x) \to \varphi(y)$. Hence φ is an epic.

Moreover, $x \in Ker\varphi \Longleftrightarrow \varphi(x) = f^1 \Longleftrightarrow f^x =_{(M,N)} f^1 \Longleftrightarrow x \sim_{f_{f(1)}} 1 \Longleftrightarrow x \in f_{f(1)}$. Hence, $Ker\varphi = f_{f(1)}$.

Thus, $L/f \cong L/f_{f(1)}$. □

5. (M,N)-SI implicative (Boolean) filters

In this section, we introduce the concept of (M,N)-SI implicative (Boolean) filters of R_0-algebras and investigate some of their properties.

Definition 8. A soft set f_L over U is called an (M,N)-soft intersection implicative filter (briefly, (M,N)-SI implicative filter) of L over U if it satisfies (SI_1) and

(SI_6) $f_L(x \to (y \to z)) \cap f_L(x \to y) \widetilde{\subseteq}_{(M,N)} f_L(x \to z)$

for all $x,y,z \in L$.

Remark 3. If f_S is an (M,N)-SI implicative filter of L over U, then f_L is an $(\emptyset,U)$-SI implicative filter of L over U. Hence, every SI-implicative filter of L is an (M,N)-SI implicative filter of L, but the converse is ont true.

Example 2. Assume that $U = D_2 = \{< x,y > |x^2 = y^2 = e, xy = yx\} = \{e,x,y,yx\}$, Dihedral group, is the universal set.

Let $L = \{0,a,b,c,d,1\}$, where $0 < a < b < c < d < 1$.

x	x'
0	1
a	d
b	c
c	b
d	a
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	1	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

Then $(L,\wedge,\vee,',\to)$ is an R_0-algebra.

Let $M = \{e,y\}$ and $N = \{e,x,y\}$.

Define a soft set f_L of L over U by $f_L(1) = \{e,x\}$, $f_L(c) = f_L(d) = \{e,x,y\}$ and $f_L(a) = f_L(b) = f_L(0) = \{e,y\}$.

Then one can easily check that f_L is an (M,N)-SI implicative filter of L over U, but it is ont an SI-implicative filter of L over U since $f_L(1) = \{e,x\} \not\supseteq f_L(c)$.

From Definitions 4 and 8, we have

Proposition 14. *Every (M,N)-SI implicative filter of L over U is an (M,N)-SI filter, but the converse may not be true as shown in the following example.*

Example 3. Consider the soft set f_L of S over U as in Example 1. Let $M = \{(13)\}$ and $N = \{(1),(12),(13),(123)\}$. We can easily check that f_L is an (M,N)-SI filter of L over U, but it is not an (M,N)-SI implicative filter of L over U since $f_L(b \to a) \cup M = f_L(b) \cup M = \{(1),(12)\} \cup \{(13)\} = \{(1),(12),(13)\} \not\supseteq \{(1),(12),(123)\} = f_L(1) \cap N = f_L(b \to (b \to a)) \cap f_L(b \to b) \cap N$.

Now, we discuss some properties of (M,N)-SI implicative filters in R_0-algebras.

Theorem 15. *Let f_L be an (M,N)-SI filter of L over U, then f_L is an (M,N)-SI implicative filter of L over U if and only if it satisfies:*

(SI_7) $f_L(x \to (z' \to y)) \cap f_L(y \to z) \widetilde{\subseteq}_{(M,N)} f_L(x \to z)$ for all $x,y,z \in L$.

Proof. *Assume that f_S is an (M,N)-SI implicative filter of L over U. For any $x,y,z \in L$, we have*

$$\begin{aligned}
&f_L(x \to z) \cup M\\
&= f_L(z' \to x') \cup M\\
&\supseteq f_L(z' \to (y' \to x')) \cap f_L(z' \to y') \cap N\\
&= f_L(x \to (z' \to y)) \cap f_L(y \to z) \cap N,
\end{aligned}$$

that is, $f_L(x \to (z' \to y)) \cap f_L(y \to z) \widetilde{\subseteq}_{(M,N)} f_L(x \to z)$. Thus, (SI_7) holds.

Conversely, assume that f_L is an (M,N)-SI filter of L over U satisfying (SI_7). Then

$$\begin{aligned}
&f_L(x \to z) \cup M\\
&= f_L(z' \to x') \cup M\\
&\supseteq f_L(z' \to (x'' \to y')) \cap f_L(y' \to x') \cap N\\
&= f_L(x \to (z' \to y')) \cap f_L(x \to y) \cap N\\
&= f_L(x \to (y \to z)) \cap f_L(x \to y) \cap N,
\end{aligned}$$

that is, $f_L(x \to (y \to z)) \cap f_L(x \to y) \widetilde{\subseteq}_{(M,N)} f_L(x \to z)$. Thus, (SI_6) holds. Therefore, f_L is an (M,N)-SI implicative filter of L over U. □

Theorem 16. *Let f_L be an (M,N)-SI filter of L over U, then the following are equivalent:*

(i) f_L is an (M,N)-SI implicative filter of L.

(ii) $f_L(x \to z) =_{(M,N)} f_L(x \to (z' \to z))$, for all $x,y,z \in L$.

(*iii*) $f_L(x \to z) \widetilde{\supseteq}_{(M,N)} f_L(y \to (x \to (z' \to z))) \cap f_L(y)$, *for all* $x,y,z \in L$.

Proof. $(i) \Rightarrow (ii)$ *Assume that* f_L *is an* (M,N)-SI *implicative filter of* L *over* U. *Putting* $y = z$ *in* (SI_7),

$$\begin{aligned} & f_L(x \to z) \cup M \\ &= (f_L(x \to z) \cup M) \cup M \\ &\supseteq (f_L(x \to (z' \to z)) \cap f_L(z \to z) \cap N) \cup M \\ &= (f_L(x \to (z' \to z)) \cap f_L(1) \cap N) \cup M \\ &\supseteq f_L(x \to (z' \to z)) \cap (f_L(1) \cup M) \cap N \\ &\supseteq f_L(x \to (z' \to z)) \cap N, \end{aligned}$$

which implies, $f_L(x \to (z' \to z)) \widetilde{\subseteq}_{(M,N)} f_L(x \to z)$.
On the other hand, $x \to z \leqslant z' \to (x \to z)$, *we have* $f_L(x \to z) \widetilde{\subseteq}_{(M,N)} f_L(z' \to (x \to z))$. *Hence,* $f_L(x \to z) =_{(M,N)} f_L(z' \to (x \to z))$.
$(ii) \Rightarrow (iii)$ *For any* $x,y,z \in L$, *we have* $f_L(x \to (z' \to z)) \widetilde{\supseteq}_{(M,N)} f_L(y \to (x \to (z' \to z))) \cap f_L(y)$.
By (ii), *we have* $f_L(x \to z) =_{(M,N)} f_L(x \to (z' \to z)) \widetilde{\supseteq}_{(M,N)} f_L(y \to (x \to (z' \to z))) \cap f_L(y)$.
Thus, (iii) *holds.*
$(iii) \Rightarrow (i)$ *Let* f_L *be an* (M,N)-SI *filter of* L *over* U *satisfying the condition* (iii). *Then by Proposition 3.7(7) for all* $x,y,z \in L$, $f_L(x \to (z' \to z)) \widetilde{\supseteq}_{(M,N)} f_L(x \to (z' \to y)) \cap f_L(y \to z)$. *Putting* $y = 1$ *in* (iii), *we have*

$$\begin{aligned} & f_L(x \to z) \cup M \\ &= (f_L(x \to z) \cup M) \cup M \\ &\supseteq (f_L(1 \to (x \to (z' \to z))) \cap f_L(1) \cap N) \cup M \\ &\supseteq (f_L(x \to (z' \to z)) \cup M) \cap (f_L(1) \cup M) \cap N \\ &\supseteq (f_L(x \to (z' \to z)) \cup M) \cap N \\ &\supseteq f_L(x \to (z' \to y)) \cap f_L(y \to z) \cap N, \end{aligned}$$

which implies, $f_L(x \to z) \widetilde{\supseteq}_{(M,N)} f_L(x \to (z' \to y)) \cap f_L(y \to z)$. *Thus,* (SI_7) *holds. By Theorem 15,* f_L *is an* (M,N)-SI *implicative filter of* L *over* U. □

Theorem 17. *Let* f_L *be an* (M,N)-SI *filter of* L *over* U, *then the following are equivalent:*
(1) f_L *is an* (M,N)-SI *implicative filter of* L.
(2) $f_L(x) =_{(M,N)} f_L(x' \to x)$, *for all* $x \in L$.
(3) $f_L(x) =_{(M,N)} f_L((x \to y) \to x)$, *for all* $x,y \in L$.
(4) $f_L(x) \widetilde{\supseteq}_{(M,N)} f_L(z \to ((x \to y) \to x)) \cap f_L(z)$, *for all* $x,y,z \in L$.

Proof. $(1) \Rightarrow (2)$ *By Theorem 16*(ii), *we have* $f_L(x) = f_L(1 \to x) =_{(M,N)} f_L(1 \to (x' \to x)) = f_L(x' \to x)$.
$(2) \Rightarrow (3)$ *Since* $x' \leqslant x \to y$, *then* $(x \to y) \to x \leqslant x' \to x$, *and so* $f_L(x' \to x) \widetilde{\supseteq}_{(M,N)} f_L((x \to y) \to x)$. *Thus, from* (2), *we can deduce that*
$f_L(x) =_{(M,N)} f_L(x' \to x) \widetilde{\supseteq}_{(M,N)} f_L((x \to y) \to x)$.
On the other hand, since $x \leqslant (x \to y) \to x$, *we have* $f_L(x) \widetilde{\subseteq}_{(M,N)} f_L((x \to y) \to x)$. *Thus, we can get* $f_L(x) =_{(M,N)} f_L((x \to y) \to x)$.
$(3) \Rightarrow (4)$ *Since* f_L *is an* (M,N)-SI *filter of* L, *then* $f_L((x \to y) \to x) \widetilde{\supseteq}_{(M,N)} f_L(z \to ((x \to y) \to x) \cap f_L(z)$. *It follows from* (3) *that*
$f_L(x) =_{(M,N)} f_L((x \to y) \to x) \widetilde{\supseteq}_{(M,N)} f_L(z \to ((x \to y) \to x) \cap f_L(z)$.
$(4) \Rightarrow (1)$ *Since* $z \leqslant x \to z$, *we have* $(x \to z)' \leqslant z'$ *and* $z' \to (x \to z) \leqslant (x \to z)' \to (x \to z)$. *Thus, we have*
$f_L((x \to z)' \to (x \to z)) \widetilde{\supseteq}_{(M,N)} f_L(z' \to (x \to z))$. *It follows from* (4) *that*

$$\begin{aligned} & f_L(x \to z) \cup M \\ &= (f_L(x \to z) \cup M) \cup M \\ &\supseteq (f_L(1 \to (((x \to z) \to 0) \to (x \to z))) \cap f_L(1) \cap N) \cup M \\ &= (f_L((x \to z)' \to (x \to z)) \cap f_L(1) \cap N) \cup M \\ &\supseteq ((f_L((x \to z)' \to (x \to z)) \cup M) \cap (f_L(1) \cup M)) \cap N \\ &\supseteq f_L(z' \to (x \to z)) \cap N, \end{aligned}$$

which implies, $f_L(x \to z) \widetilde{\supseteq}_{(M,N)} f_L(z' \to (x \to z))$.
On the other hand, since $x \to z \leqslant z' \to (x \to z)$, *we have* $f_L(x \to z) \widetilde{\subseteq}_{(M,N)} f_L(z' \to (x \to z))$. *Thus,* $f_L(x \to z) =_{(M,N)} f_L(z' \to (x \to z))$. *Therefore, it follows from Theorem 16 that* f_L *is an* (M,N)-SI *implicative filter of* L *over* U. □

Finally, we introduce the concept of (M,N)-SI Boolean filters of R_0-algebras.

Definition 9. Let f_L be an (M,N)-SI filter of L over U, then f_L is called an (M,N)-SI Boolean filter of L if it satisfies
(SI_8) $f_L(x \vee x') =_{(M,N)} f_L(1)$, for all $x \in L$.

Theorem 18. *A soft set* f_L *over* U *is an* (M,N)-SI *implicative filter of* L *if and only if it is an* (M,N)-SI *Boolean filter.*

Proof. *Assume that* f_L *is an* (M,N)-SI *implicative filter of* L *over* U. *For any* $x \in L$, *since* $x' \to (((x' \to x) \to x) \to (x' \to x)') = ((x' \to x) \to x) \to ((x' \to x) \to x) = 1$ *and* $x' \to ((x' \to x) \to x) = 1$, *then*

$$f_L((x' \to x) \to x)$$
$$= f_L(x' \to (x' \to x)')$$
$$\widetilde{\supseteq}_{(M,N)} f(x' \to (((x' \to x) \to x) \to (x' \to x)'))$$
$$\cap f_L(x' \to ((x' \to x) \to x))$$
$$= f_L(1),$$

and so, $f_L((x' \to x) \to x) =_{(M,N)} f_L(1)$.
Similarly, we can prove $f_L((x \to x') \to x') =_{(M,N)} f_L(1)$. *Hence, we have*

$$f_L(x \vee x')$$
$$= f_L(((x' \to x) \to x) \wedge ((x \to x') \to x'))$$
$$=_{(M,N)} f_L((x' \to x) \to x) \cap f_L((x \to x') \to x')$$
$$=_{(M,N)} f_L(1).$$

This proves that f_L *is an* (M,N)-SI *Boolean filter of* L.

Conversely, assume that f_L *is an* (M,N)-SI *Boolean filter of* L. *For any* $x, y \in L$, *we have*

$$f_L(x \to y) \cup M$$
$$= (f_L(x \to y) \cup M) \cup M$$
$$\supseteq (f_L((y \vee y') \to (x \to y)) \cap f_L(y \vee y') \cap N) \cup M$$
$$= (f_L((y \vee y') \to (x \to y)) \cap f_L(1) \cap N) \cup M$$
$$\supseteq (f_L((y \vee y') \to (x \to y)) \cup M) \cap N$$
$$= ((f_L(y \to (x \to y)) \cap f_L(y' \to (x \to y))) \cup M) \cap N$$
$$= (f_L(1) \cup M) \cap (f_L(y' \to (x \to y)) \cup M) \cap N$$
$$\supseteq f_L(y' \to (x \to y)) \cap N,$$

which implies, $f_L(x \to y) \widetilde{\supseteq}_{(M,N)} f_L(y' \to (x \to y))$.

On the other hand, since $x \to y \leqslant x \to (y' \to y)$, *we have* $f_L(x \to y) =_{(M,N)} f_L(x \to (y' \to y))$.

Therefore, it follows from Theorem 16 that f_L *is an* (M,N)-SI *implicative filter of* L. □

Remark 4. Every (M,N)-SI implicative filters and (M,N)-SI Boolean filters in R_0-algebras are equivalent.

6. Conclusion

As a generalization of soft intersection filters of R_0-algebras, we introduce the concepts of (M,N)-SI (implicative) filters of R_0-algebras. We investigate their characterizations. In particular, we describe (M,N)-soft congruences in R_0-algebras.

To extend this work, one can further investigate (M,N)-SI prime (semiprime) filters of R_0-algebras. Maybe one can apply this idea to decision making, data analysis and knowledge based systems.

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