A Note on "A Lexicographic Method for Matrix Games with Payoffs of Triangular Intuitionistic Fuzzy Numbers"

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Abstract

Nan et al. [J.-X. Nan, D.-F. Li and M.-J. Zhang, A lexicographic method for matrix games with payoffs of triangular intuitionistic fuzzy numbers, International Journal of Computational Intelligence Systems 3(3) (2010) 280-289] pointed out that there is no method in the literature to find the solution of such matrix games in which payoffs are represented by triangular intuitionistic fuzzy numbers and proposed a method for the same. In this paper, it is pointed out that Nan et al. have used some mathematical incorrect assumptions in their proposed method and the existing method is also modified.

Keywords: Triangular intuitionistic fuzzy number; Intuitionistic fuzzy set; Matrix game; Mathematical programming; Lexicographic method.

1. Existing Method for Comparing Triangular Intuitionistic Fuzzy Numbers

In this section, the method, used by Nan et al. [1] for comparing triangular intuitionistic fuzzy numbers, is presented [1, Section 2.2, Definition 7, pp. 283].

If $\tilde{a}_1 = \left\langle \left(\underline{a}_1, a_1, \overline{a}_1\right); w_{\tilde{a}_1}, u_{\tilde{a}_1} \right\rangle$ and $\tilde{a}_2 = \left\langle \left(\underline{a}_2, a_2, \overline{a}_2\right); w_{\tilde{a}_2}, u_{\tilde{a}_2} \right\rangle$ are two triangular intuitionistic fuzzy numbers. Then,

i. $\tilde{a}_1 >_{IF} \tilde{a}_2$ if $S_\mu(\tilde{a}_1) > S_\mu(\tilde{a}_2)$ or if $S_\mu(\tilde{a}_1) = S_\mu(\tilde{a}_2)$ then $S_\upsilon(\tilde{a}_1) > S_\upsilon(\tilde{a}_2)$.

ii. $\tilde{a}_1 =_{IF} \tilde{a}_2$ if and only if $S_\mu(\tilde{a}_1) = S_\mu(\tilde{a}_2)$ and $S_\upsilon(\tilde{a}_1) = S_\upsilon(\tilde{a}_2)$.

where, $S_\mu\left(\tilde{a}_i\right) = \dfrac{\left(w_{\tilde{a}_i}\right)\left(2a_i + \underline{a}_i + \overline{a}_i\right)}{4}$ and $S_\upsilon\left(\tilde{a}_i\right) = \dfrac{\left(1 - u_{\tilde{a}_i}\right)\left(2a_i + \underline{a}_i + \overline{a}_i\right)}{4}$, are called membership and non-membership function average indexes respectively.

2. Existing Method

Nan et al. [1] proposed the following method to find the optimal solution (maximin strategy for player 1, minimax strategy for player 2 and value of intuitionistic fuzzy matrix game for player 1) of such intuitionistic

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fuzzy matrix games in which payoffs are represented by triangular intuitionistic fuzzy numbers.

Step 1: Formulate the chosen problem as the mathematical programming problems P1 and P2 to find the maximin strategy $y_i^*, i=1,2,\ldots m$ and minimax strategy $z_j^*, j=1,2,\ldots n$ for player 1 and player 2 respectively.

Problem P1 [1, Equation 3, pp. 284]

Maximize$(\tilde{v})$

Subject to

$$\sum_{i=1}^{m}\tilde{a}_{ij}y_i \geq_{IF} \tilde{v},\ j=1,2,\ldots,n;$$
$$\sum_{i=1}^{m} y_i = 1;$$
$$y_i \geq 0,\ i=1,2,\ldots,m.$$

Problem P2 [1, Equation 4, pp. 284]

Minimize$(\tilde{\omega})$

Subject to

$$\sum_{j=1}^{n}\tilde{a}_{ij}z_j \leq_{IF} \tilde{\omega},\ i=1,2,\ldots,m;$$
$$\sum_{j=1}^{n} z_j = 1;$$
$$z_j \geq 0,\ j=1,2,\ldots,n.$$

Step 2: Using Section 1, the problems P1 and P2 can be transformed into problems P3 and P4 respectively.

Problem P3 [1, Equation 5, pp. 284]

Maximize$\{S_\mu(\tilde{v}), S_\upsilon(\tilde{v})\}$

Subject to

$$S_\mu\left(\sum_{i=1}^{m}\tilde{a}_{ij}y_i\right) \geq S_\mu(\tilde{v}), \qquad j=1,2,\ldots,n;$$
$$S_\upsilon\left(\sum_{i=1}^{m}\tilde{a}_{ij}y_i\right) \geq S_\upsilon(\tilde{v}), \qquad j=1,2,\ldots,n;$$
$$\sum_{i=1}^{m} y_i = 1;$$
$$y_i \geq 0, i=1,2,\ldots,m.$$

Problem P4 [1, Equation 10, pp. 286]

Minimize$\{S_\mu(\tilde{\omega}), S_\upsilon(\tilde{\omega})\}$

Subject to

$$S_\mu\left(\sum_{j=1}^{n}\tilde{a}_{ij}z_j\right) \leq S_\mu(\tilde{\omega}), i=1,2,\ldots,m;$$
$$S_\upsilon\left(\sum_{j=1}^{n}\tilde{a}_{ij}z_j\right) \leq S_\upsilon(\tilde{\omega}), i=1,2,\ldots,m;$$
$$\sum_{j=1}^{n} z_j = 1;$$
$$z_j \geq 0, j=1,2,\ldots,n.$$

Step 3: Using the properties $S_\mu\left(\sum_{i=1}^{n}\tilde{a}_i\right)=\sum_{i=1}^{n}S_\mu(\tilde{a}_i)$ and $S_\upsilon\left(\sum_{i=1}^{n}\tilde{a}_i\right)=\sum_{i=1}^{n}S_\upsilon(\tilde{a}_i)$, the problems P3 and P4 can be transformed into problems P5 and P6 respectively.

Problem P5 [1, Equation 6, pp. 285]

Maximize$\{S_\mu(\tilde{v}), S_\upsilon(\tilde{v})\}$

Subject to

$$\sum_{i=1}^{m} S_\mu(\tilde{a}_{ij}y_i) \geq S_\mu(\tilde{v}), j=1,2,\ldots,n;$$
$$\sum_{i=1}^{m} S_\upsilon(\tilde{a}_{ij}y_i) \geq S_\upsilon(\tilde{v}), j=1,2,\ldots,n;$$
$$\sum_{i=1}^{m} y_i = 1;$$
$$y_i \geq 0, i=1,2,\ldots,m.$$

Problem P6 [1, Equation 11, pp. 286]

Minimize$\{S_\mu(\tilde{\omega}), S_\upsilon(\tilde{\omega})\}$

Subject to

$$\sum_{j=1}^{n} S_\mu(\tilde{a}_{ij}z_j) \leq S_\mu(\tilde{\omega}), i=1,2,\ldots,m;$$
$$\sum_{j=1}^{n} S_\upsilon(\tilde{a}_{ij}z_j) \leq S_\upsilon(\tilde{\omega}), i=1,2,\ldots,m;$$
$$\sum_{j=1}^{n} z_j = 1;$$
$$z_j \geq 0, j=1,2,\ldots,n.$$

Step 4: Putting $S_\mu(\tilde{a})=\frac{(w_{\tilde{a}})(2a+\underline{a}+\overline{a})}{4}$ and $S_\upsilon(\tilde{a})=\frac{(1-u_{\tilde{a}})(2a+\underline{a}+\overline{a})}{4}$, the problems P5 and P6 can be transformed into problems P7 and P8 respectively.

Problem P7 [1, Equation 6, pp. 285]

Maximize$\left\{(w_{\tilde{v}})\frac{(\underline{v}+2v+\overline{v})}{4}, (1-u_{\tilde{v}})\frac{(\underline{v}+2v+\overline{v})}{4}\right\}$

Subject to

$$\sum_{i=1}^{m}\frac{\left(w_{\tilde{a}_{ij}}\right)\left(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij}\right)y_i}{4}\geq\left(w_{\tilde{v}}\right)\frac{\left(\underline{v}+2v+\overline{v}\right)}{4},\ j=1,2,\ldots n;$$

$$\sum_{i=1}^{m}\frac{\left(1-u_{\tilde{a}_{ij}}\right)\left(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij}\right)y_i}{4}\geq\left(1-u_{\tilde{v}}\right)\frac{\left(\underline{v}+2v+\overline{v}\right)}{4},\ j=1,2,\ldots n;$$

$\underline{v}\leq v$;

$v\leq\overline{v}$;

$\sum_{i=1}^{m}y_i=1$;

$y_i\geq 0,\quad i=1,2,\ldots,m.$

where, $w_{\tilde{v}}=\min_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{w_{\tilde{a}_{ij}}\right\}$ and $u_{\tilde{v}}=\max_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{u_{\tilde{a}_{ij}}\right\}$.

Problem P8 [1, Equation 11, pp. 286]

Minimize $\left\{\left(w_{\tilde{\omega}}\right)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4},\left(1-u_{\tilde{\omega}}\right)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4}\right\}$

Subject to

$$\sum_{j=1}^{n}\frac{\left(w_{\tilde{a}_{ij}}\right)\left(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij}\right)z_j}{4}\leq\left(w_{\tilde{\omega}}\right)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4},i=1,2,\ldots m;$$

$$\sum_{j=1}^{n}\frac{\left(1-u_{\tilde{a}_{ij}}\right)\left(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij}\right)z_j}{4}\leq\left(1-u_{\tilde{\omega}}\right)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4},i=1,2,\ldots m;$$

$\underline{\omega}\leq\omega$;

$\omega\leq\overline{\omega}$;

$\sum_{j=1}^{n}z_j=1$;

$z_j\geq 0,\quad j=1,2,\ldots,n.$

where, $w_{\tilde{\omega}}=\min_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{w_{\tilde{a}_{ij}}\right\}$ and $u_{\tilde{\omega}}=\max_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{u_{\tilde{a}_{ij}}\right\}$.

Step 5: Assuming $\left(w_{\tilde{v}}\right)\frac{\left(\underline{v}+2v+\overline{v}\right)}{4}=v_1,\left(1-u_{\tilde{v}}\right)\frac{\left(\underline{v}+2v+\overline{v}\right)}{4}=v_2$ and $\left(w_{\tilde{\omega}}\right)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4}=\omega_1$, $\left(1-u_{\tilde{\omega}}\right)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4}=\omega_2$, the problems P7 and P8 can be transformed into problems P9 and P10 respectively.

Problem P9 [1, Equation 7, pp. 285]

Maximize $\left\{v_1,v_2\right\}$

Subject to

$$\sum_{i=1}^{m}\left(\frac{\left(w_{\tilde{a}_{ij}}\right)\left(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij}\right)y_i}{4}\right)\geq v_1,\ j=1,2,\ldots n;$$

$$\sum_{i=1}^{m}\left(\frac{\left(1-u_{\tilde{a}_{ij}}\right)\left(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij}\right)y_i}{4}\right)\geq v_2,\ j=1,2,\ldots n;$$

$v_2\geq v_1$;

$\sum_{i=1}^{m}y_i=1$;

$y_i\geq 0, i=1,2,\ldots,m.$

Problem P10 [1, Equation 12, pp. 286]

Minimize $\left\{\omega_1,\omega_2\right\}$

Subject to

$$\sum_{j=1}^{n}\left(\frac{\left(w_{\tilde{a}_{ij}}\right)\left(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij}\right)z_j}{4}\right)\leq\omega_1, i=1,2,\ldots m;$$

$$\sum_{j=1}^{n}\left(\frac{\left(1-u_{\tilde{a}_{ij}}\right)\left(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij}\right)z_j}{4}\right)\leq\omega_2, i=1,2,\ldots m;$$

$\omega_1\leq\omega_2$;

$\sum_{j=1}^{n}z_j=1$;

$z_j\geq 0,\quad j=1,2,\ldots,n.$

Step 6: Using lexicographic method the optimal solution of problems P9 and P10 and hence the optimal solution of problems P1 and P2 respectively can be obtained as follows.

Step 6(a): Find the optimal solution $\left\{v_1,v_2,y_i,i=1,2,\ldots,m\right\}$ and $\left\{\omega_1,\omega_2,z_j,j=1,2,\ldots,n\right\}$ of problems P11 and P12 respectively. Let it be denoted by $\left\{v_1^0,v_2^0,y_i^0,i=1,2,\ldots,m\right\}$ and $\left\{\omega_1^0,\omega_2^0,z_j^0,j=1,2,\ldots,n\right\}$ respectively.

Problem P11 [1, Equation 8, pp. 285]

Maximize $\left\{v_1\right\}$

Subject to

Constraints of problem P9.

Problem P12 [1, Equation 13, pp. 286]

Minimize $\left\{\omega_1\right\}$

Subject to

Constraints of problem P10.

Step 6(b): Find the optimal solution

$\{\nu_1,\nu_2,y_i,i=1,2,\ldots,m\}$ and $\{\omega_1,\omega_2,z_j,j=1,2,\ldots,n\}$ of problems P13 and P14 respectively. Let it be denoted by $\{\nu_1^*,\nu_2^*,y_i^*,i=1,2,\ldots,m\}$ and $\{\omega_1^*,\omega_2^*,z_j^*,j=1,2,\ldots,n\}$ respectively.

Problem P13 [1, Equation 9, pp. 285]

Maximize $\{\nu_2\}$

Subject to

Constraints of problem P9 with additional constraints

$\nu_1 \geq \nu_1^0;$

$\nu_2 \geq \nu_2^0.$

Problem P14 [1, Equation 14, pp. 286]

Minimize $\{\omega_2\}$

Subject to

Constraints of problem P10 with additional constraints

$\omega_1 \leq \omega_1^0;$

$\omega_2 \leq \omega_2^0.$

Step 7: Using the maximin strategy $y_i^*, i=1,2,\ldots m$ and minimax strategy $z_j^*, j=1,2,\ldots n$, obtained in Step 6(b), the value of intuitionistic fuzzy matrix game for player1 is $E\left(y_i^*,z_j^*\right)=\sum_{i=1}^{m}\sum_{j=1}^{n} y_i^*\tilde{a}_{ij}z_j^*.$

3. Error in the Existing Method

If $\tilde{a}_i=\left\langle\left(\underline{a}_i,a_i,\overline{a}_i\right);w_{\tilde{a}_i},u_{\tilde{a}_i}\right\rangle$, $i=1,2,\ldots,n$ are n triangular intuitionistic fuzzy numbers then using Definition 2 [1, Section 2.1, pp. 282],

$$\text{(i)}\sum_{i=1}^{n}\tilde{a}_i=\sum_{i=1}^{n}\left(\left\langle\left(\underline{a}_i,a_i,\overline{a}_i\right);w_{\tilde{a}_i},u_{\tilde{a}_i}\right\rangle\right)$$
$$=\left\langle\left(\sum_{i=1}^{n}\underline{a}_i,\sum_{i=1}^{n}a_i,\sum_{i=1}^{n}\overline{a}_i\right);\min_{1\leq i\leq n}(w_{\tilde{a}_i}),\max_{1\leq i\leq n}(u_{\tilde{a}_i})\right\rangle.$$

$$\text{(ii)}\,\lambda\tilde{a}_i=\lambda\left\langle\left(\underline{a}_i,a_i,\overline{a}_i\right);w_{a_i},u_{a_i}\right\rangle=$$
$$=\begin{cases}\left\langle\left(\lambda\underline{a}_i,\lambda a_i,\lambda\overline{a}_i\right);w_{\tilde{a}_i},u_{\tilde{a}_i}\right\rangle, & \lambda\geq 0,\\ \left\langle\left(\lambda\overline{a}_i,\lambda a_i,\lambda\underline{a}_i\right);w_{\tilde{a}_i},u_{\tilde{a}_i}\right\rangle, & \lambda<0.\end{cases}$$

Therefore,

$$S_\mu\left(\sum_{i=1}^{n}\tilde{a}_i\right)=\min_{1\leq i\leq n}\left(w_{\tilde{a}_i}\right)\frac{\left(\sum_{i=1}^{n}\underline{a}_i+2\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}\overline{a}_i\right)}{4},$$

$$S_\upsilon\left(\sum_{i=1}^{n}\tilde{a}_i\right)=\left(1-\max_{1\leq i\leq n}\left(u_{\tilde{a}_i}\right)\right)\frac{\left(\sum_{i=1}^{n}\underline{a}_i+2\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}\overline{a}_i\right)}{4}$$

and $\sum_{i=1}^{n}S_\mu(\tilde{a}_i)=\sum_{i=1}^{n}\frac{\left(w_{\tilde{a}_i}\right)\left(\underline{a}_i+2a_i+\overline{a}_i\right)}{4}$,

$$\sum_{i=1}^{n}S_\upsilon(\tilde{a}_i)=\sum_{i=1}^{n}\frac{(1-u_{\tilde{a}_i})\left(\underline{a}_i+2a_i+\overline{a}_i\right)}{4}.$$

It is obvious that $S_\mu\left(\sum_{i=1}^{n}\tilde{a}_i\right)\neq\sum_{i=1}^{n}S_\mu\left(\tilde{a}_i\right)$

and $S_\upsilon\left(\sum_{i=1}^{n}\tilde{a}_i\right)\neq\sum_{i=1}^{n}S_\upsilon\left(\tilde{a}_i\right)$.

However, in Step 3 of the existing method, described in Section 2, for transforming problems P3 and P4 into problems P5 and P6 respectively, Nan et al. [1] have used the mathematical properties $S_\mu\left(\sum_{i=1}^{m}\tilde{a}_{ij}y_i\right)=\sum_{i=1}^{m}S_\mu\left(\tilde{a}_{ij}y_i\right)$, $S_\upsilon\left(\sum_{i=1}^{m}\tilde{a}_{ij}y_i\right)=\sum_{i=1}^{m}S_\upsilon\left(\tilde{a}_{ij}y_i\right)$ and

$S_\mu\left(\sum_{j=1}^{n}\tilde{a}_{ij}z_j\right)=\sum_{j=1}^{n}S_\mu\left(\tilde{a}_{ij}z_j\right)$, $S_\upsilon\left(\sum_{j=1}^{n}\tilde{a}_{ij}z_j\right)=\sum_{j=1}^{n}S_\upsilon\left(\tilde{a}_{ij}z_j\right)$

respectively, which do not hold in general.

Hence, there is error in the existing method [1].

4. Modified Method

In this section, the existing method [1] is modified and the steps are as follows.

Step 1: Assuming $\tilde{a}_{ij}=\left\langle\left(\underline{a}_{ij},a_{ij},\overline{a}_{ij}\right);w_{\tilde{a}_{ij}},u_{\tilde{a}_{ij}}\right\rangle$, $\tilde{\nu}=\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle$ and $\tilde{\omega}=\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle$, where $w_{\tilde{\nu}}=w_{\tilde{\omega}}=\min_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{w_{\tilde{a}_{ij}}\right\}$ and $u_{\tilde{\nu}}=u_{\tilde{\omega}}=\max_{\substack{1\leq i\leq m\\1\leq j\leq n}}\left\{u_{\tilde{a}_{ij}}\right\}$, the problems P1 and P2 can be transformed into problems P15 and P16 respectively.

Problem P15

Maximize$\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle\right)$

Subject to

$$\sum_{i=1}^{m}\left\langle\left(\underline{a}_{ij},a_{ij},\overline{a}_{ij}\right);w_{\tilde{a}_{ij}},u_{\tilde{a}_{ij}}\right\rangle y_i\geq_{IF}\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle,\ j=1,2,\ldots,n;$$

$$\sum_{i=1}^{m}y_i=1;$$

$y_i\geq 0,\ i=1,2,\ldots,m.$

Problem P16

Minimize$\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle\right)$

Subject to

$$\sum_{j=1}^{n}\left\langle\left(\underline{a}_{ij},a_{ij},\overline{a}_{ij}\right);w_{\tilde{a}_{ij}},u_{\tilde{a}_{ij}}\right\rangle z_j\leq_{IF}\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle,\ i=1,2,\ldots,m;$$
$$\sum_{j=1}^{n}z_j=1;$$
$$z_j\geq 0,\ j=1,2,\ldots,n.$$

Step 2: Using the arithmetic operation $\left\langle\left(\underline{a},a,\overline{a}\right);w_{\tilde{a}},u_{\tilde{a}}\right\rangle\lambda=\left\langle\left(\underline{a}\lambda,a\lambda,\overline{a}\lambda\right);w_{\tilde{a}},u_{\tilde{a}}\right\rangle,\lambda\geq 0$, the problems P15 and P16 can be transformed into problems P17 and P18 respectively.

Problem P17

Maximize $\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle\right)$

Subject to

$$\sum_{i=1}^{m}\left\langle\left(\underline{a}_{ij}y_i,a_{ij}y_i,\overline{a}_{ij}y_i\right);w_{\tilde{a}_{ij}},u_{\tilde{a}_{ij}}\right\rangle\geq_{IF}\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle,\ j=1,2,\ldots,n;$$
$$\sum_{i=1}^{m}y_i=1;$$
$$y_i\geq 0,\ i=1,2,\ldots,m.$$

Problem P18

Minimize $\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle\right)$

Subject to

$$\sum_{j=1}^{n}\left\langle\left(\underline{a}_{ij}z_j,a_{ij}z_j,\overline{a}_{ij}z_j\right);w_{\tilde{a}_{ij}},u_{\tilde{a}_{ij}}\right\rangle\leq_{IF}\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle,\ i=1,2,\ldots,m;$$
$$\sum_{j=1}^{n}z_j=1;$$
$$z_j\geq 0,\ j=1,2,\ldots,n.$$

Step 3: Using the arithmetic operation

$$\sum_{i=1}^{n}\left\langle\left(\underline{a}_i,a_i,\overline{a}_i\right);w_{\tilde{a}_i},u_{\tilde{a}_i}\right\rangle=\left\langle\left(\sum_{i=1}^{n}\underline{a}_i,\sum_{i=1}^{n}a_i,\sum_{i=1}^{n}\overline{a}_i\right);\min_{1\leq i\leq n}\left(w_{\tilde{a}_i}\right),\max_{1\leq i\leq n}\left(u_{\tilde{a}_i}\right)\right\rangle$$

, the problems P17 and P18 can be transformed into problems P19 and P20 respectively.

Problem P19

Maximize $\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle\right)$

Subject to

$$\left\langle\left(\sum_{i=1}^{m}\underline{a}_{ij}y_i,\sum_{i=1}^{m}a_{ij}y_i,\sum_{i=1}^{m}\overline{a}_{ij}y_i\right);\min_{1\leq i\leq m}\left(w_{\tilde{a}_{ij}}\right),\max_{1\leq i\leq m}\left(u_{\tilde{a}_{ij}}\right)\right\rangle\geq_{IF}\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle,\ j=1,2,\ldots,n;$$
$$\sum_{i=1}^{m}y_i=1;$$
$$y_i\geq 0,\ i=1,2,\ldots,m.$$

Problem P20

Minimize $\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle$

Subject to

$$\left\langle\left(\sum_{j=1}^{n}\underline{a}_{ij}z_j,\sum_{j=1}^{n}a_{ij}z_j,\sum_{j=1}^{n}\overline{a}_{ij}z_j\right);\min_{1\leq j\leq n}\left(w_{\tilde{a}_{ij}}\right),\max_{1\leq j\leq n}\left(u_{\tilde{a}_{ij}}\right)\right\rangle\leq_{IF}\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle,\ i=1,2,\ldots,m;$$
$$\sum_{j=1}^{n}z_j=1;$$
$$z_j\geq 0,\ j=1,2,\ldots,n.$$

Step 4: Using Section 1, the problems P19 and P20 can be transformed into problems P21 and P22 respectively.

Problem P21

Maximize

$$\left\{S_\mu\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle\right),S_\upsilon\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle\right)\right\}$$

Subject to

$$S_\mu\left(\left\langle\left(\sum_{i=1}^{m}\underline{a}_{ij}y_i,\sum_{i=1}^{m}a_{ij}y_i,\sum_{i=1}^{m}\overline{a}_{ij}y_i\right);\min_{1\leq i\leq m}\left(w_{\tilde{a}_{ij}}\right),\max_{1\leq i\leq m}\left(u_{\tilde{a}_{ij}}\right)\right\rangle\right)\geq S_\mu\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle\right),\ j=1,2,\ldots,n;$$
$$S_\upsilon\left(\left\langle\left(\sum_{i=1}^{m}\underline{a}_{ij}y_i,\sum_{i=1}^{m}a_{ij}y_i,\sum_{i=1}^{m}\overline{a}_{ij}y_i\right);\min_{1\leq i\leq m}\left(w_{\tilde{a}_{ij}}\right),\max_{1\leq i\leq m}\left(u_{\tilde{a}_{ij}}\right)\right\rangle\right)\geq S_\upsilon\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);w_{\tilde{\nu}},u_{\tilde{\nu}}\right\rangle\right),\ j=1,2,\ldots,n;$$
$$\sum_{i=1}^{m}y_i=1;$$
$$y_i\geq 0,\ i=1,2,\ldots,m.$$

Problem P22

Minimize

$$\left\{S_\mu\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle\right),S_\upsilon\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle\right)\right\}$$

Subject to

$$S_\mu\left(\left\langle\left(\sum_{j=1}^{n}\underline{a}_{ij}z_j,\sum_{j=1}^{n}a_{ij}z_j,\sum_{j=1}^{n}\overline{a}_{ij}z_j\right);\min_{1\leq j\leq n}\left(w_{\tilde{a}_{ij}}\right),\max_{1\leq j\leq n}\left(u_{\tilde{a}_{ij}}\right)\right\rangle\right)\leq S_\mu\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle\right),\ i=1,2,\ldots,m;$$

$$S_\upsilon\left(\left\langle\left(\sum_{j=1}^{n}\underline{a}_{ij}z_j,\sum_{j=1}^{n}a_{ij}z_j,\sum_{j=1}^{n}\overline{a}_{ij}z_j\right);\min_{1\le j\le n}\left(w_{\tilde{a}_{ij}}\right),\max_{1\le j\le n}\left(u_{\tilde{a}_{ij}}\right)\right\rangle\right)\le S_\upsilon\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);w_{\tilde{\omega}},u_{\tilde{\omega}}\right\rangle\right),\ i=1,2,\ldots,m;$$

$$\sum_{j=1}^{n}z_j=1;$$

$$z_j\ge 0,\ j=1,2,\ldots,n.$$

Step 5: Using Section 1, the problems P21 and P22 can be transformed into problems P23 and P24 respectively.

Problem P23

$$\text{Maximize}\left\{\left(w_{\tilde{\nu}}\right)\frac{\left(\underline{\nu}+2\nu+\overline{\nu}\right)}{4},\left(1-u_{\tilde{\nu}}\right)\frac{\left(\underline{\nu}+2\nu+\overline{\nu}\right)}{4}\right\}$$

Subject to

$$\left(\min_{1\le i\le m}\left(w_{\tilde{a}_{ij}}\right)\right)\frac{\left(\sum_{i=1}^{m}\underline{a}_{ij}y_i+2\sum_{i=1}^{m}a_{ij}y_i+\sum_{i=1}^{m}\overline{a}_{ij}y_i\right)}{4}\ge\left(w_{\tilde{\nu}}\right)\frac{\left(\underline{\nu}+2\nu+\overline{\nu}\right)}{4},\ j=1,2,\ldots,n;$$

$$\left(1-\max_{1\le i\le m}\left(u_{\tilde{a}_{ij}}\right)\right)\frac{\left(\sum_{i=1}^{m}\underline{a}_{ij}y_i+2\sum_{i=1}^{m}a_{ij}y_i+\sum_{i=1}^{m}\overline{a}_{ij}y_i\right)}{4}\ge\left(1-u_{\tilde{\nu}}\right)\frac{\left(\underline{\nu}+2\nu+\overline{\nu}\right)}{4},\ j=1,2,\ldots,n;$$

$$\underline{\nu}\le\nu;$$

$$\nu\le\overline{\nu};$$

$$\sum_{i=1}^{m}y_i=1;$$

$$y_i\ge 0,\ i=1,2,\ldots,m.$$

Problem P24

$$\text{Minimize}\left\{\left(w_{\tilde{\omega}}\right)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4},\left(1-u_{\tilde{\omega}}\right)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4}\right\}$$

Subject to

$$\left(\min_{1\le j\le n}\left(w_{\tilde{a}_{ij}}\right)\right)\frac{\left(\sum_{j=1}^{n}\underline{a}_{ij}z_j+2\sum_{j=1}^{n}a_{ij}z_j+\sum_{j=1}^{n}\overline{a}_{ij}z_j\right)}{4}\le\left(w_{\tilde{\omega}}\right)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4},\ i=1,2,\ldots,m;$$

$$\left(1-\max_{1\le j\le n}\left(u_{\tilde{a}_{ij}}\right)\right)\frac{\left(\sum_{j=1}^{n}\underline{a}_{ij}z_j+2\sum_{j=1}^{n}a_{ij}z_j+\sum_{j=1}^{n}\overline{a}_{ij}z_j\right)}{4}\le\left(1-u_{\tilde{\omega}}\right)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4},\ i=1,2,\ldots,m;$$

$$\underline{\omega}\le\omega;$$

$$\omega\le\overline{\omega};$$

$$\sum_{j=1}^{n}z_j=1;$$

$$z_j\ge 0,\ j=1,2,\ldots,n.$$

Step 6: Assuming $\left(w_{\tilde{\nu}}\right)\left(\underline{\nu}+2\nu+\overline{\nu}\right)/4=\nu_1$, $\left(1-u_{\tilde{\nu}}\right)\left(\underline{\nu}+2\nu+\overline{\nu}\right)/4=\nu_2$ and $\left(w_{\tilde{\omega}}\right)\left(\underline{\omega}+2\omega+\overline{\omega}\right)/4=\omega_1$, $\left(1-u_{\tilde{\omega}}\right)\left(\underline{\omega}+2\omega+\overline{\omega}\right)/4=\omega_2$ the problems P23 and P24 can be transformed into problems P25 and P26 respectively.

Problem P25

$$\text{Maximize}\left\{\nu_1,\nu_2\right\}$$

Subject to

$$\left(\min_{1\le i\le m}\left(w_{\tilde{a}_{ij}}\right)\right)\frac{\left(\sum_{i=1}^{m}\underline{a}_{ij}y_i+2\sum_{i=1}^{m}a_{ij}y_i+\sum_{i=1}^{m}\overline{a}_{ij}y_i\right)}{4}\ge\nu_1,\ j=1,2,\ldots,n;$$

$$\left(1-\max_{1\le i\le m}\left(u_{\tilde{a}_{ij}}\right)\right)\frac{\left(\sum_{i=1}^{m}\underline{a}_{ij}y_i+2\sum_{i=1}^{m}a_{ij}y_i+\sum_{i=1}^{m}\overline{a}_{ij}y_i\right)}{4}\ge\nu_2,\ j=1,2,\ldots,n;$$

$$\nu_2\ge\nu_1;$$

$$\sum_{i=1}^{m}y_i=1;$$

$$y_i\ge 0,\ i=1,2,\ldots,m.$$

Problem P26

$$\text{Minimize}\left\{\omega_1,\omega_2\right\}$$

Subject to

$$\left(\min_{1\le j\le n}\left(w_{\tilde{a}_{ij}}\right)\right)\frac{\left(\sum_{j=1}^{n}\underline{a}_{ij}z_j+2\sum_{j=1}^{n}a_{ij}z_j+\sum_{j=1}^{n}\overline{a}_{ij}z_j\right)}{4}\le\omega_1,\ i=1,2,\ldots,m;$$

$$\left(1-\max_{1\le j\le n}\left(u_{\tilde{a}_{ij}}\right)\right)\frac{\left(\sum_{j=1}^{n}\underline{a}_{ij}z_j+2\sum_{j=1}^{n}a_{ij}z_j+\sum_{j=1}^{n}\overline{a}_{ij}z_j\right)}{4}\le\omega_2,\ i=1,2,\ldots,m;$$

$$\omega_1\le\omega_2;$$

$$\sum_{j=1}^{n}z_j=1;$$

$$z_j\ge 0,\ j=1,2,\ldots,n.$$

Step 7: Using lexicographic method the optimal solution of the problems P25 and P26 and hence the optimal solution of problems P1 and P2 respectively can be obtained as follows.

Step 7(a): Find the optimal solution $\{\nu_1,\nu_2,y_i,i=1,2,\ldots,m\}$ and $\{\omega_1,\omega_2,z_j,j=1,2,\ldots,n\}$ of problems P27 and P28. Let it be denoted by $\{\nu_1^0,\nu_2^0,y_i^0,i=1,2,\ldots,m\}$ and $\{\omega_1^0,\omega_2^0,z_j^0,j=1,2,\ldots,n\}$ respectively.

Problem P27
Maximize $\{\nu_1\}$
Subject to
Constraints of problem P25.

Problem P28
Minimize $\{\omega_1\}$
Subject to
Constraints of problem P26.

Step 7(b): Find the optimal solution $\{\nu_1,\nu_2,y_i,i=1,2,\ldots,m\}$ and $\{\omega_1,\omega_2,z_j,j=1,2,\ldots,n\}$ of problems P29 and P30 respectively. Let it be denoted by $\{\nu_1^*,\nu_2^*,y_i^*,i=1,2,\ldots,m\}$ and $\{\omega_1^*,\omega_2^*,z_j^*,j=1,2,\ldots,n\}$ respectively.

Problem P29
Maximize $\{\nu_2\}$
Subject to
Constraints of problem P25 with additional constraints
$\nu_1\ge\nu_1^0;$
$\nu_2\ge\nu_2^0.$

Problem P30
Minimize $\{\omega_2\}$
Subject to
Constraints of problem P26 with additional constraints
$\omega_1\le\omega_1^0;$
$\omega_2\le\omega_2^0.$

Step 8: Using the maximin strategy $y_i^*,i=1,2,\ldots m$ and minimax strategy $z_j^*,j=1,2,\ldots n$, obtained in Step 7(b), the value of intuitionistic fuzzy matrix game for player1 is $E(y_i^*,z_j^*)=\sum_{i=1}^{m}\sum_{j=1}^{n}y_i^*\tilde{a}_{ij}z_j^*.$

5. Exact Optimal Solution of Market Share Problem

Nan et al. [1, Section 4, pp. 287] solved market share problem to illustrate their proposed method by considering two companies p_1 and p_2 and

$$\tilde{A}=\begin{pmatrix}\langle(175,180,190);0.6,0.2\rangle & \langle(150,156,158);0.6,0.1\rangle\\ \langle(80,90,100);0.9,0.1\rangle & \langle(175,180,190);0.6,0.2\rangle\end{pmatrix}$$

as intuitionistic fuzzy payoffs matrix for company p_1, where p_1 and p_2 are regarded as player 1 and player 2 respectively. However, as discussed in Section 3, that Nan et al. [1] have used some mathematical incorrect assumptions. So, the intuitionistic fuzzy optimal solution of market share problem, obtained by Nan et al. [1], is not exact.

In this section, to find the exact optimal solution (maximin strategy $y_i^*,i=1,2,\ldots m$, minimax strategy $z_j^*,j=1,2,\ldots n$ for player 1, player 2 respectively and value of intuitionistic fuzzy matrix game for player 1) of this problem, the corresponding intuitionistic fuzzy linear programming problems P31 and P32, is solved by the modified method.

Problem P31
Maximize $(\tilde{\nu})$
Subject to
$\langle(175,180,190);0.6,0.2\rangle y_1+\langle(80,90,100);0.9,0.1\rangle y_2\ge_{IF}\tilde{\nu};$
$\langle(150,156,158);0.6,0.1\rangle y_1+\langle(175,180,190);0.6,0.2\rangle y_2\ge_{IF}\tilde{\nu};$
$y_1+y_2=1;$
$y_1,y_2\ge 0.$

Problem P32
Minimize $(\tilde{\omega})$
Subject to
$\langle(175,180,190);0.6,0.2\rangle z_1+\langle(150,156,158);0.6,0.1\rangle z_2\le_{IF}\tilde{\omega};$
$\langle(80,90,100);0.9,0.1\rangle z_1+\langle(175,180,190);0.6,0.2\rangle z_2\le_{IF}\tilde{\omega};$
$z_1+z_2=1;$
$z_1,z_2\ge 0.$

Using modified method problems P31 and P32 can be solved as follows:

Step 1: Assuming $\tilde{\nu}=\langle(\underline{\nu},\nu,\overline{\nu});0.6,0.2\rangle$ and $\tilde{\omega}=\langle(\underline{\omega},\omega,\overline{\omega});0.6,0.2\rangle$, where $0.6=\min_{\substack{1\le i\le 2\\1\le j\le 2}}\{w_{\tilde{a}_{ij}}\}=\{0.6,0.6,0.9,0.6\}$ and $0.2=\max_{\substack{1\le i\le 2\\1\le j\le 2}}\{u_{\tilde{a}_{ij}}\}=\{0.2,0.1,0.1,0.2\}$, the problems P31 and P32 can be transformed into problems P33 and P34 respectively.

Problem P33

Maximize $\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle\right)$

Subject to

$$\left\langle(175,180,190);0.6,0.2\right\rangle y_1+\left\langle(80,90,100);0.9,0.1\right\rangle y_2 \geq_{IF} \left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle;$$

$$\left\langle(150,156,158);0.6,0.1\right\rangle y_1+\left\langle(175,180,190);0.6,0.2\right\rangle y_2 \geq_{IF} \left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle;$$

$y_1+y_2=1;$

$y_1,y_2\geq 0.$

Problem P34

Minimize $\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle\right)$

Subject to

$$\left\langle(175,180,190);0.6,0.2\right\rangle z_1+\left\langle(150,156,158);0.6,0.1\right\rangle z_2 \leq_{IF} \left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle;$$

$$\left\langle(80,90,100);0.9,0.1\right\rangle z_1+\left\langle(175,180,190);0.6,0.2\right\rangle z_2 \leq_{IF} \left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle;$$

$z_1+z_2=1;$

$z_1,z_2\geq 0.$

Step 2: Using the arithmetic operation $\left\langle\left(\underline{a},a,\overline{a}\right);w_{\tilde{a}},u_{\tilde{a}}\right\rangle\lambda=\left\langle\left(\underline{a}\lambda,a\lambda,\overline{a}\lambda\right);w_{\tilde{a}},u_{\tilde{a}}\right\rangle,\lambda\geq 0;$ the problems P33 and P34 can be transformed into problems P35 and P36 respectively.

Problem P35

Maximize $\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle\right)$

Subject to

$$\begin{pmatrix}\left\langle(175y_1,180y_1,190y_1);0.6,0.2\right\rangle+\\ \left\langle(80y_2,90y_2,100y_2);0.9,0.1\right\rangle\end{pmatrix}\geq_{IF}\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle;$$

$$\begin{pmatrix}\left\langle(150y_1,156y_1,158y_1);0.6,0.1\right\rangle+\\ \left\langle(175y_2,180y_2,190y_2);0.6,0.2\right\rangle\end{pmatrix}\geq_{IF}\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle;$$

$y_1+y_2=1;$

$y_1,y_2\geq 0.$

Problem P36

Minimize $\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle\right)$

Subject to

$$\begin{pmatrix}\left\langle(175z_1,180z_1,190z_1);0.6,0.2\right\rangle+\\ \left\langle(150z_2,156z_2,158z_2);0.6,0.1\right\rangle\end{pmatrix}\leq_{IF}\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle;$$

$$\begin{pmatrix}\left\langle(80z_1,90z_1,100z_1);0.9,0.1\right\rangle+\\ \left\langle(175z_2,180z_2,190z_2);0.6,0.2\right\rangle\end{pmatrix}\leq_{IF}\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle;$$

$z_1+z_2=1;$

$z_1,z_2\geq 0.$

Step 3: Using the arithmetic operation $\sum_{i=1}^{n}\left\langle\left(\underline{a}_i,a_i,\overline{a}_i\right);w_{\tilde{a}_i},u_{\tilde{a}_i}\right\rangle=\left\langle\left(\sum_{i=1}^{n}\underline{a}_i,\sum_{i=1}^{n}a_i,\sum_{i=1}^{n}\overline{a}_i\right);\min_{1\leq i\leq n}\left(w_{\tilde{a}_i}\right),\max_{1\leq i\leq n}\left(u_{\tilde{a}_i}\right)\right\rangle$,the problems P35 and P36 can be transformed into problems P37 and P38 respectively.

Problem P37

Maximize $\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle\right)$

Subject to

$$\left\langle(175y_1+80y_2,180y_1+90y_2,190y_1+100y_2);0.6,0.2\right\rangle\geq_{IF}\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle;$$

$$\left\langle(150y_1+175y_2,156y_1+180y_2,158y_1+190y_2);0.6,0.2\right\rangle\geq_{IF}\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle;$$

$y_1+y_2=1;$

$y_1,y_2\geq 0.$

Problem P38

Minimize $\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle\right)$

Subject to

$$\left\langle(175z_1+150z_2,180z_1+156z_2,190z_1+158z_2);0.6,0.2\right\rangle\leq_{IF}\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle;$$

$$\left\langle(80z_1+175z_2,90z_1+180z_2,100z_1+190z_2);0.6,0.2\right\rangle\leq_{IF}\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle;$$

$z_1+z_2=1;$

$z_1,z_2\geq 0.$

Step 4: Using Section 1, the problems P37 and P38 can be transformed into problems P39 and P40 respectively.

Problem P39

Maximize

$$\left\{S_\mu\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle\right),S_\upsilon\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle\right)\right\}$$

Subject to

$$S_\mu\left(\left\langle\left(175y_1+80y_2,180y_1+90y_2,190y_1+100y_2\right);0.6,0.2\right\rangle\right)\geq S_\mu\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle\right);$$
$$S_\upsilon\left(\left\langle\left(175y_1+80y_2,180y_1+90y_2,190y_1+100y_2\right);0.6,0.2\right\rangle\right)\geq S_\upsilon\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle\right);$$
$$S_\mu\left(\left\langle\left(150y_1+175y_2,156y_1+180y_2,158y_1+190y_2\right);0.6,0.2\right\rangle\right)\geq S_\mu\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle\right);$$
$$S_\upsilon\left(\left\langle\left(150y_1+175y_2,156y_1+180y_2,158y_1+190y_2\right);0.6,0.2\right\rangle\right)\geq S_\upsilon\left(\left\langle\left(\underline{\nu},\nu,\overline{\nu}\right);0.6,0.2\right\rangle\right);$$
$$y_1+y_2=1;$$
$$y_1,y_2\geq 0.$$

Problem P40

Minimize

$$\left\{S_\mu\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle\right),S_\upsilon\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle\right)\right\}$$

Subject to

$$S_\mu\left(\left\langle\left(175z_1+150z_2,180z_1+156z_2,190z_1+158z_2\right);0.6,0.2\right\rangle\right)\leq S_\mu\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle\right);$$
$$S_\upsilon\left(\left\langle\left(175z_1+150z_2,180z_1+156z_2,190z_1+158z_2\right);0.6,0.2\right\rangle\right)\leq S_\upsilon\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle\right);$$
$$S_\mu\left(\left\langle\left(80z_1+175z_2,90z_1+180z_2,100z_1+190z_2\right);0.6,0.2\right\rangle\right)\leq S_\mu\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle\right);$$
$$S_\upsilon\left(\left\langle\left(80z_1+175z_2,90z_1+180z_2,100z_1+190z_2\right);0.6,0.2\right\rangle\right)\leq S_\upsilon\left(\left\langle\left(\underline{\omega},\omega,\overline{\omega}\right);0.6,0.2\right\rangle\right);$$
$$z_1+z_2=1;$$
$$z_1,z_2\geq 0.$$

Step 5: Using Section 1, the problems P39 and P40 can be transformed into problems P41 and P42 respectively.

Problem P41

$$\text{Maximize}\left\{(0.6)\frac{\left(\underline{\nu}+2\nu+\overline{\nu}\right)}{4},(1-0.2)\frac{\left(\underline{\nu}+2\nu+\overline{\nu}\right)}{4}\right\}$$

Subject to

$$(0.6)\frac{\left(175y_1+80y_2+2\left(180y_1+90y_2\right)+190y_1+100y_2\right)}{4}\geq(0.6)\frac{\left(\underline{\nu}+2\nu+\overline{\nu}\right)}{4};$$
$$(1-0.2)\frac{\left(175y_1+80y_2+2\left(180y_1+90y_2\right)+190y_1+100y_2\right)}{4}\geq(1-0.2)\frac{\left(\underline{\nu}+2\nu+\overline{\nu}\right)}{4};$$
$$(0.6)\frac{\left(150y_1+175y_2+2\left(156y_1+180y_2\right)+158y_1+190y_2\right)}{4}\geq(0.6)\frac{\left(\underline{\nu}+2\nu+\overline{\nu}\right)}{4};$$
$$(1-0.2)\frac{\left(150y_1+175y_2+2\left(156y_1+180y_2\right)+158y_1+190y_2\right)}{4}\geq(1-0.2)\frac{\left(\underline{\nu}+2\nu+\overline{\nu}\right)}{4};$$
$$\underline{\nu}\leq\nu;$$
$$\nu\leq\overline{\nu};$$
$$y_1+y_2=1;$$
$$y_1,y_2\geq 0.$$

Problem P42

$$\text{Minimize}\left\{(0.6)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4},(1-0.2)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4}\right\}$$

Subject to

$$(0.6)\frac{\left(175z_1+150z_2+2\left(180z_1+156z_2\right)+190z_1+158z_2\right)}{4}\leq(0.6)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4};$$
$$(1-0.2)\frac{\left(175z_1+150z_2+2\left(180z_1+156z_2\right)+190z_1+158z_2\right)}{4}\leq(1-0.2)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4};$$
$$(0.6)\frac{\left(80z_1+175z_2+2\left(90z_1+180z_2\right)+100z_1+190z_2\right)}{4}\leq(0.6)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4};$$
$$(1-0.2)\frac{\left(80z_1+175z_2+2\left(90z_1+180z_2\right)+100z_1+190z_2\right)}{4}\leq(1-0.2)\frac{\left(\underline{\omega}+2\omega+\overline{\omega}\right)}{4};$$
$$\underline{\omega}\leq\omega;$$
$$\omega\leq\overline{\omega};$$
$$z_1+z_2=1;$$
$$z_1,z_2\geq 0.$$

Step 6: Assuming $(0.6)\left(\underline{\nu}+2\nu+\overline{\nu}\right)/4=\nu_1$, $(1-0.2)\left(\underline{\nu}+2\nu+\overline{\nu}\right)/4=\nu_2$ and $(0.6)\left(\underline{\omega}+2\omega+\overline{\omega}\right)/4=\omega_1$,

$\left(1-0.2\right)\left(\underline{\omega}+2\omega+\overline{\omega}\right)/4=\omega_2$, the problems P41 and P42 can be transformed into problems P43 and P44 respectively.

Problem P43

Maximize $\{\nu_1,\nu_2\}$

Subject to

$$(0.6)\frac{\left(175y_1+80y_2+2\left(180y_1+90y_2\right)+190y_1+100y_2\right)}{4}\geq \nu_1;$$

$$(1-0.2)\frac{\left(175y_1+80y_2+2\left(180y_1+90y_2\right)+190y_1+100y_2\right)}{4}\geq \nu_2;$$

$$(0.6)\frac{\left(150y_1+175y_2+2\left(156y_1+180y_2\right)+158y_1+190y_2\right)}{4}\geq \nu_1;$$

$$(1-0.2)\frac{\left(150y_1+175y_2+2\left(156y_1+180y_2\right)+158y_1+190y_2\right)}{4}\geq \nu_2;$$

$\nu_2\geq \nu_1;$

$y_1+y_2=1;$

$y_1,y_2\geq 0.$

Problem P44

Minimize $\{\omega_1,\omega_2\}$

Subject to

$$(0.6)\frac{\left(175z_1+150z_2+2\left(180z_1+156z_2\right)+190z_1+158z_2\right)}{4}\leq \omega_1;$$

$$(1-0.2)\frac{\left(175z_1+150z_2+2\left(180z_1+156z_2\right)+190z_1+158z_2\right)}{4}\leq \omega_2;$$

$$(0.6)\frac{\left(80z_1+175z_2+2\left(90z_1+180z_2\right)+100z_1+190z_2\right)}{4}\leq \omega_1;$$

$$(1-0.2)\frac{\left(80z_1+175z_2+2\left(90z_1+180z_2\right)+100z_1+190z_2\right)}{4}\leq \omega_2;$$

$\omega_1\leq \omega_2;$

$z_1+z_2=1;$

$z_1,z_2\geq 0.$

Step 7: Using lexicographic method the optimal solution of problems P43 and P44 and hence the optimal solution of problems P31 and P32 respectively can be obtained as follows.

Step 7(a): The optimal solution of problems P45 and P46 is $\left\{\nu_1=\frac{36291}{376},\nu_2=\frac{36291}{376},y_1=\frac{73}{94},y_2=\frac{21}{94}\right\}$ and $\left\{\omega_1=\frac{36291}{376},\omega_2=\frac{12097}{94},z_1=\frac{21}{94},z_2=\frac{73}{94}\right\}$ respectively.

Problem P45

Maximize $\{\nu_1\}$

Subject to

Constraints of problem P43.

Problem P46

Minimize $\{\omega_1\}$

Subject to

Constraints of problem P44.

Step 7 (b): The exact optimal solution of problems P47 and P48 and hence of problems P31 and P32 is $\left\{\nu_1=\frac{36291}{376},\nu_2=\frac{12097}{94},y_1=\frac{73}{94},y_2=\frac{21}{94}\right\}$ and $\left\{\omega_1=\frac{36291}{376},\omega_2=\frac{12097}{94},z_1=\frac{21}{94},z_2=\frac{73}{94}\right\}$ respectively.

Problem P47

Maximize $\{\nu_2\}$

Subject to

Constraints of problem P43 with additional constraints

$\nu_1\geq \frac{36291}{376};$

$\nu_2\geq \frac{36291}{376}.$

Problem P48

Maximize $\{\omega_2\}$

Subject to

Constraints of problem P44 with additional constraints

$\omega_1\leq \frac{36291}{376};$

$\omega_2\leq \frac{12097}{94}.$

Step 8: Using the maximin strategy $y_1^*=\frac{73}{94},y_2^*=\frac{21}{94}$ and minimax strategy $z_1^*=\frac{21}{94},z_2^*=\frac{73}{94}$, obtained in Step 7(b), the exact solution of intuitionistic fuzzy matrix game for the market share problem of company p_1 is

$$E\left(y_i^*, z_j^*\right) = \left\langle \left(\frac{342795}{2209}, \frac{711447}{4418}, \frac{734311}{4418} \right); 0.6, 0.2 \right\rangle$$

$= \left\langle (155.18, 161.03, 166.21); 0.6, 0.2 \right\rangle$ as shown in the following Fig. 1.

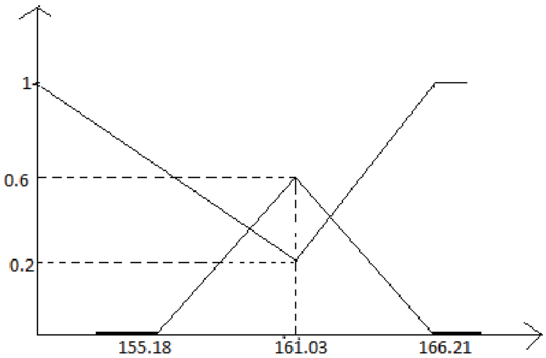

Fig.1: The exact solution of intuitionistic fuzzy matrix game for the market share problem of company p_1

6. Conclusion

The error in the existing method [1] is pointed out and the modified method is proposed to find the exact solution of such matrix games in which payoffs are represented by triangular intuitionistic fuzzy numbers.

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