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Understanding and Visualization of the Uniform Continuity of Functions

Ljubica Dikovic[1, a]

[1]Business-Technical College, Trg Svetog Save 34, 31000 Uzice, Serbia.

[a]dikoviclj@gmail.com

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Abstract. The formal definition of the concept of continuity, due to its dynamic essence, is perfectly suited to visual representation by software tools. This paper presents the classical teaching approach supported by GeoGebra, for teaching and learning very specific and subtle criteria which distinguish concept uniform continuity of functions compared to the concept of continuity. Without this software tool, in the graphic terms, it would not be possible that students make the distinction between sophisticated class uniformly continuous and non-uniformly continuous functions. Also, the contribution of this paper is presentation some specifics examples to better understanding the concept of continuity vs. uniform continuity at the college levels on the interactive and visual way.

Introduction

The representation process includes the use of different models for organizing, memorizing and exchanging of math ideas with the aim of solving math problems and for a better interpretation of mathematics.

Representation, graphic and otherwise, in the function of reasoning has been explored by many researchers. Research on learning with representations has shown that when students interact with an appropriate representation their performance is enhanced. Computer algebra systems can be used to change the emphasis on learning and teaching of calculus away from symbolic techniques and methods towards higher-level cognitive skills that focus on concepts and problem-solving. ([1, 2, 3, 4, 6, 7,11]).

Mathematical concepts, ideas, methods, have significant support in technology that is intuitively representative in different ways. The use of them is very beneficial in the process of learning and teaching when solving problems and doing research. The research literature confirms that technology can enhance students' understanding of mathematics concepts ([1, 3, 4, 8, 10,11]) and improve their achievement. Learning calculus has been subject of extensive research for a long time. The research literature indicates that students have experienced cognitive difficulties in understanding the concept of limit, which is the key concept of mathematical analysis ([1, 2, 3]).

Visualization of Continuity and Uniform Continuity

In the calculus, where continuity of functions is one of the core concepts, definition of continuity should cover two ideas: Graphically, the graph of f is a smooth curve with no jumps, gaps, or holes, and the second, the values of a function $f(x)$ at points near x_0 tend to $f(x_0)$.

Informally, a continuous function is a function for which small changes in the input bring small changes in the output of that function.

Formally, there are several definitions of continuity which can be used to determine whether a given function is continuous or not.

Function $y = f(x)$ is continuous at point $x = x_0$ if the following three conditions are satisfied:

$$\text{(i) } f(x_0) \text{ is defined, (ii) } \lim_{x\to x_0} f(x) \text{ exists (i.e., is finite), and (iii) } \lim_{x\to x_0} f(x) = f(x_0). \qquad (1)$$

Cauchy-Weierstrass definition (epsilon-delta, or formal definition) of continuity read as follows: A function $f: D \to R$ on the real domain $D \subset R$ is continuous at the point x_0 if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D)\left(|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon\right). \quad (2)$$

This Cauchy-Weierstrass definition of continuity of a function $f(x)$ implies very complex levels of the layers of quantifiers and is a concept that is very difficult to understand.

It was showed, that GeoGebra has many possibilities to help students to get an intuitive feeling and to visualize adequate math process. The use of this software's tools allows students to explore a wider range of function types and provides students to make the connections between symbolic and visual representations. [10]

The function $f: D \to R$ defined on an interval $(a,b) \subset D$ is said to be uniformly continuous in this interval, if for $(\forall \varepsilon > 0)$ there is $(\delta = \delta(\varepsilon) > 0)$, such that for all pairs of points $x_1, x_2 \in (a,b)$ for which $|x_2 - x_1| < \delta$, $|f(x_2) - f(x_1)| < \varepsilon$ is true.

By comparing these two definitions is obtained:

- Continuity: For every $x_0 \in A$ and for any $\varepsilon > 0$, there is some $\delta > 0$ such that $f(x)$ is within ε of $f(x_0)$ whenever x is within δ of x_0.
- Uniform continuity: For any, $\varepsilon > 0$ there is some $\delta > 0$ such that for every $x_0 \in A$, $f(x)$ is within ε of $f(x_0)$ whenever x is within δ of x_0.

The difference between uniform continuity and continuity is that continuity of a function is purely a local property - for a fixed x_0, if x within δ of x_0, then $f(x)$ is within ε of $f(x_0)$ (for appropriate δ and ε), whereas uniform continuity is a global property that applies to the whole space - for any x_0, if x is within δ of x_0, then $f(x)$ is within ε of $f(x_0)$, where δ and ε are independent of the choice of x_0. All uniformly continuous curves are continuous, but the reverse does not apply.

The sudden increase (decrease) in curves is not possible in uniform continuity, whereas if there are no gaps in this curve with the possibility of sudden increase (decrease), then the curve is simply continuous. This sudden increase (decrease) disrupts the uniformity of continuous curves (see figure 2). As x approaches 0 from the right side, the curve increases exponentially, which is not the case of uniformity.

Since the no break in the function of the interval (0, 1), we conclude that the function is continuous, but not uniformly continuous.

In what follows it was presented a few examples that help to perceive how visualization may be of great help in the teaching and learning continuity and later on it were presented some others which are a little more sophisticated – uniform continuity.

This provides the following graphical representation created with GeoGebra (for example, for function $y = x^2 + 2x$, see figure 1.):

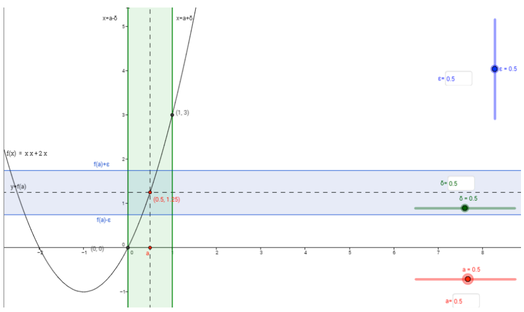

Fig. 1Epsilon-delta definition, example created with GeoGebra

In order to provide understanding the notion of continuity of arbitrary function $f(x)$ firstly introduces the concept of a rectangular box that is centered on the arbitrary points of the graph of $f(x)$, with a height of 2ε and width of 2δ, and which sides are parallel to the axes Ox, Oy.

To visualize continuity: given any point $(x_0, f(x_0))$ and any choice of box height ε, there is a box width δ so that the graph of the function "leaves" the box centered at $(x_0, f(x_0))$ through its left and right sides. Uniform continuity means that there is a choice of δ that will allow sliding this box of the graph of the function without the possibility that top or bottom of the box ever intersecting the graph, i.e. there is one δ that will satisfy uniformly with all locations of ε.

With visualizing, we can conclude that a uniformly continuous function is actually a continuous function that changes in a controlled way. In this sense, uniform continuity is a tool used to determine how uniformly behaved a continuous function is. For functions defined on a closed interval, uniform continuity is equivalent to continuity.

The function $g(x) = \frac{1}{x}$ is continuous on the open interval (0, 1). Is it uniformly continuous there?

It helps to look at the graph of the function (applet) created with GeoGebra: In the graphical sense, as the ε - interval slides up with a positive end of y-axis, the corresponding δ -interval on the x-axis gets smaller and smaller which indicates that the function is not uniformly continuous (see figure 2). If we take values of x and y that are arbitrarily close to zero, then we will have to make δ smaller and smaller, but $f(x)$ and $f(y)$ would not be within ε of each other. For uniform continuity, there has to be one single δ that satisfies with any possible location of the ε interval of the y-axis. In the picture below that is not possible.

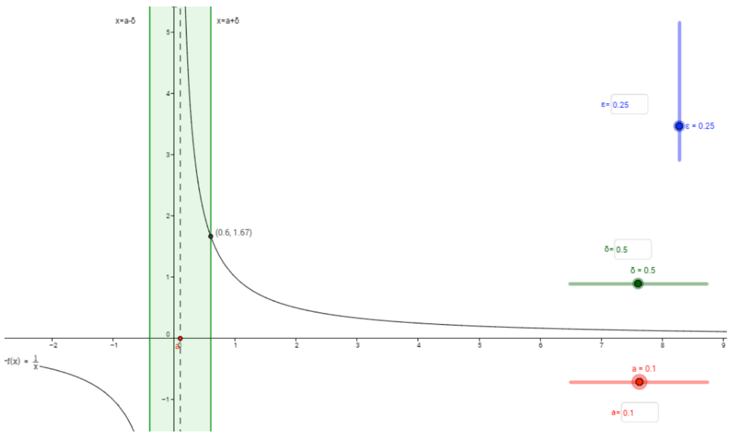

Fig. 2 A Function that is not uniformly continuous on (0, 1), example created with GeoGebra

Summary

The focus of the paper is to devise activities that will prepare college students to use technology to enhance and extend their students' learning of mathematics.

If we incorporate the right combination of available software tools of the classical teaching of mathematics, we can achieve that the students develop and strengthen the informal and intuitive understanding of certain mathematical concept, which would lead to substantial and deeper understanding of the subject matter, and provides students to make the connections between symbolic and visual representations.

The formal definition of the concept of continuity, due to its dynamic essence, is perfectly suited to visual representation by software tools. Given the fact that the formal definition of uniform continuity

is different from the definition of continuity in a mathematically subtle way, it would be desirable that each teacher designs appropriate software environment and exercises, in order to further motivate the students to thoroughly understand these already complex and abstract concepts.

References

[1] Hohenwarter, M. Hohenwarter, J., Kreis, Y., Lavicza, Z., Teaching and learning calculus with free dynamic mathematics software GeoGebra. Proceedings of the 11th International Congress on Mathematics Education. The university of Nuevo Leon, Monterrey, Mexico. (2008)

[2] Tall,D., Katz,M., A Cognitive Analysis of Cauchy's Conceptions of Function, Continuity, Limit, and Infinitesimal, with implications for teaching calculus. Educational Studies in Mathematics. 86 (1) (2014) 97-124.

[3] Cappetta, Robert W., Zollman, A., Agents of Change in Promoting Reflective Abstraction: A Quasi-Experimental, Study on Limits in College Calculus. Journal of Research in Mathematics Education. 2 (3) (2013) 343-357.

[4] Borchelt, N., Cognitive Computer Tools in the Teaching and Learning of Undergraduate Calculus, International Journal for the Scholarship of Teaching and Learning. 1 (2) (2007) 12.

[5] Guzman, M.: The Role of Visualization in the Teaching and Learning of Mathematical Analysis. Proceedings of the International Conference on the Teaching of Mathematics at the Undergraduate Level, Publisher ERIC, Greece.(2002)

[6] Özmantar, M., Akkoç H., Bingölbali E., Demir S., Ergene B. : Pre-Service Mathematics Teachers'Use of Multiple Representations in Technology-Rich Environments, Eurasia Journal of Mathematics, Vol. 6, No. 1, 19-36. (2010)

[7] Ainsworth, S.: DeFT: a conceptual framework for considering learning with multiple representations, Journal of Learning and Instruction. Vol. 16, 183-198. (2006)

[8] Kersaint, G.: Technology Beliefs and Practices of Mathematics Education Faculty. Journal of Technology and Teacher Education, 11(4) (2003) 549-577.

[9] Arcavi, A., The role of visual representations in the learning of mathematics, Educational Studies in Mathematics, 52 (2003) 215-241.

[10] Diković, L.: Applications GeoGebra into Teaching Some Topics of Mathematics at the College Level. Computer Science and Information Systems, 6 (2) (2009) 191-203.

[11] Tall, D., Dynamic mathematics and the blending of knowledge structures in the calculus. ZDM – The International Journal of Mathematics Education. 41(4) (2009) 481–492.

[12] Information on http://www.mathcs.org/analysis/reals/cont/answers/contunif.html

[13] www.quora.com/What-is-the-difference-between-continuous-and-uniformly-continuous-for-a-function

[14] Information on https://www.geogebra.org/