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Persistence of Global Well-Posedness for Fractional Dissipation Boussinesq System

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***Abstract*—The goal of this paper is to address the global existence and the uniqueness of solution to the 2D fractional dissipation Boussinesq system. And further prove the persitence in the space $H^{1+s}(R^2)\times H^{1+s}(R^2)$, $s\in(0,1)$.**

Keywords-existence; uniqueness; fractional dissipation; Boussinesq system

I. INTRODUCTION

In this paper, we study the Cauchy problem for the 2D Boussinesq equations with fractional dissipation. The model readsas follow

$$\begin{cases} u_t+\nu\Lambda^{2\alpha}u+u\cdot\nabla u+\nabla P=\theta e_2, \\ \operatorname{div} u=0, \\ \theta_t+\kappa\Lambda^{2\beta}\theta+u\cdot\nabla\theta=0, \\ u(x,0)=u_0(x),\ \theta(x,0)=\theta_0(x). \end{cases} \tag{1}$$

where $u=(u_1,u_2)$ is the velocity vector field. θ and P denote the scalar temperature and pressure of the fluid, respectively. The positive constants ν and κ denote the viscosity and thermal diffusivity. $e_2=(0,1)$ is the unit vector in the vertical direction. For the sake of simplicity, we denote $\Lambda:=\sqrt{-\Delta}$, the square root of the negative Laplacian.

The Boussinesq system is nonlinear partial differential equations models the thermal convection and geophysical flows, which plays an important role in the atmospheric sciences and oceanographic turbulence (see, e.g., [1, 2]). Our main focus of the research on the 2D Boussinesq system has been on the global regularity issue when only fractional dissipation is present. So far, there has been a lot of literature about this model.

II. MAIN RESULT

The following is the main result of this paper which asserts the the global existence and the uniqueness of solution to the 2D fractional dissipation Boussinesq system (1).

Theorem 1 Let $\nu>0,\kappa>0$, $\alpha,\beta\in\left(\frac{2}{3},1\right)$ and $\alpha+\beta>\frac{2}{3}$. Assume that $(u_0,\theta_0)\in H^{1+s}\times H^{1+s}, s\in(0,1)$. Then there exists a unique global solution (u,θ) of the Boussinesq system (1), such that, for any $T>0$,

$$\begin{cases} u\in C([0,T];H^{1+s})\bigcap L^2(0,T;H^{1+s+\alpha}), \\ \theta\in C([0,T];H^{1+s})\bigcap L^2(0,T;H^{1+s+\beta}). \end{cases}$$

III. PROOF OF THEOREM

The proof is divided into two main parts: the global existence and uniqueness.

A. The Proof of Gglobal Existence

Taking L^2 inner product of the third equation of (1) with θ, we can get

$$\frac{d}{dt}\|\theta\|^2+2\kappa\left\|\Lambda^\beta\theta\right\|^2=0 \tag{2}$$

Similarly, from the firest equation of (1) we know that

$$\begin{aligned} \frac{d}{dt}\|u\|^2+2\nu\left\|\Lambda^\alpha u\right\|^2 &= \int\theta e_2\cdot udx \\ &\le\frac{1}{2}\|\theta\|^2+\frac{1}{2}\|u\|^2. \end{aligned} \tag{3}$$

Summing up (2) and (3), we have

$$\frac{d}{dt}\left(\|u\|^2+\|\theta\|^2\right)+2\nu\left\|\Lambda^\alpha u\right\|^2+2\kappa\left\|\Lambda^\beta\theta\right\|^2\le\frac{1}{2}\|\theta\|^2+\frac{1}{2}\|u\|^2. \tag{4}$$

Applying the Gronwall inequality to (4), we can get, for all $t\in[0,T]$

$$\|u\|^2+\|\theta\|^2+2\nu\int_0^t\left\|\Lambda^\alpha u\right\|^2 d\tau+2\kappa\int_0^t\left\|\Lambda^\beta\theta\right\|^2 d\tau\le C,\quad (5)$$

where $C=C(\|u_0\|,\|\theta_0\|,T)$ is a positive constant.

Next, taking the curl of the Boussinesq system (1), we obtain that

$$\omega_t+\nu\Lambda^{2\alpha}\omega+u\cdot\nabla\omega=\theta_{x_1}\ ,\quad (6)$$

where $\omega=\partial_{x_1}u_2-\partial_{x_2}u_1$. Multiplying (6) by ω and interating the result equation by parts, we have

$$\begin{aligned}\frac{d}{dt}\|\omega\|^2+2\nu\left\|\Lambda^\alpha\omega\right\|^2&=\int\theta_{x_1}\cdot\omega dx\\ &\le\frac{1}{2\nu}\left\|\Lambda^{1-\alpha}\theta\right\|^2+\frac{\nu}{2}\left\|\Lambda^\alpha u\right\|^2.\end{aligned}\quad (7)$$

By the condition, and use the emmbedding inequality, it follows that

$$\frac{d}{dt}\|\omega\|^2+\nu\left\|\Lambda^\alpha\omega\right\|^2\le\frac{1}{\nu}\|\nabla\theta\|^2.\quad (8)$$

Taking L^2 inner product of the third equation of (1) with $\Delta\theta$, we obtain

$$\frac{d}{dt}\|\nabla\theta\|^2+2\kappa\left\|\Lambda^{1+\beta}\theta\right\|^2=0.\quad (9)$$

Summing (8) and (9) tigether, and applying the Gronwall inequality, we can get, for all $t\in[0,T]$

$$\|\omega\|^2+\|\nabla\theta\|^2+\nu\int_0^t\left\|\Lambda^\alpha\omega\right\|^2 d\tau+2\kappa\int_0^t\left\|\Lambda^{1+\beta}\theta\right\|^2 d\tau\le C,\quad (10)$$

where $C=C(\|u_0\|_{H^1},\|\theta_0\|_{H^1},T)$ is a positive constant.

Taking L^2 inner product of the third equation of (1) with $\Lambda^{2+2s}\theta$, we obtain

$$\frac{d}{dt}\left\|\Lambda^{1+s}\theta\right\|^2+2\kappa\left\|\Lambda^{1+s+\beta}\theta\right\|^2=-2\int(u\cdot\nabla\theta)\cdot\Lambda^{2+2s}\theta dx.\quad (11)$$

Since u is divergence free, then $u\cdot\nabla\theta=\nabla\cdot(u\theta)$, using the Kate-Ponce inequality from [3] (see also, e.g., [4, 5]), we have

$$\begin{aligned}&-\int(u\cdot\nabla\theta)\cdot\Lambda^{2+2s}\theta dx\\ &\le\left|\int(u\cdot\nabla\theta)\cdot\Lambda^{2+2s}\theta dx\right|\\ &\le C\left(\left\|\Lambda^{2+s-\beta}u\right\|_{L^4}\|\theta\|_{L^4}+\|u\|_{L^6}\left\|\Lambda^{2+s-\beta}\theta\right\|_{L^3}\right)\left\|\Lambda^{1+s+\beta}\theta\right\|.\end{aligned}\quad (12)$$

Applying the fractional embedding theorems and the Young inequality, we arrive at

$$\begin{aligned}&-\int(u\cdot\nabla\theta)\cdot\Lambda^{2+2s}\theta dx\\ &\le\frac{\nu}{4}\left\|\Lambda^{1+s+\alpha}u\right\|^2+\frac{\kappa}{2}\left\|\Lambda^{1+s+\beta}\theta\right\|^2+F(\|u\|,\|\theta\|,\|\nabla u\|,\|\nabla\theta\|),\end{aligned}\quad (13)$$

where

$$F=C\left(\|\theta\|^{\frac{2(s+\alpha)}{2\alpha+2\beta-3}}\|\nabla\theta\|^{\frac{2(s+\alpha)}{2\alpha+2\beta-3}}\|\nabla u\|^2+\|\nabla\theta\|^2\|u\|^{\frac{s+\beta}{3\beta-2}}\|\nabla u\|^{\frac{2s+2\beta}{3\beta-2}}\right)$$

is an explicit polynomial. Inserting (13) into (11), we can get

$$\begin{aligned}&\frac{d}{dt}\left\|\Lambda^{1+s}\theta\right\|^2+\kappa\left\|\Lambda^{1+s+\beta}\theta\right\|^2\\ &\le\frac{\nu}{2}\left\|\Lambda^{1+s+\alpha}u\right\|^2+F(\|u\|,\|\theta\|,\|\nabla u\|,\|\nabla\theta\|).\end{aligned}\quad (14)$$

Applying the operator Λ^{1+s} to the first equation of (1), and taking L^2 inner product with $\Lambda^{1+s}u$, then, integrating by parts, we obtain

$$\begin{aligned}&\frac{d}{dt}\left\|\Lambda^{1+s}u\right\|^2+2\nu\left\|\Lambda^{1+s+\alpha}u\right\|^2\\ &=-2\int\Lambda^{1+s}(u_j\partial_j u_k)\cdot\Lambda^{1+s}u_k dx+2\int\Lambda^{1+s}(\theta e_2)\Lambda^{1+s}u dx.\end{aligned}\quad (15)$$

Using the Kate-Ponce inequality again, and applying the fractional embedding theorems and the Young inequality, we can get

$$\begin{aligned}&-\int\Lambda^{1+s}(u_j\partial_j u_k)\cdot\Lambda^{1+s}u_k dx\\ &\le\left|\int\Lambda^{1+s}(u_j\partial_j u_k)\cdot\Lambda^{1+s}u_k dx\right|\\ &\le C\left(\left\|\Lambda^{1+s-\alpha}u\right\|_{L^4}\|\nabla u\|_{L^4}+\|u\|_{L^6}\left\|\Lambda^{2+s-\alpha}u\right\|_{L^3}\right)\left\|\Lambda^{1+s+\alpha}u\right\|\\ &\le\frac{\nu}{4}\left\|\Lambda^{1+s+\alpha}u\right\|^2+C\left\|\Lambda^{1+s}u\right\|^2+G(\|u\|,\|\nabla u\|),\end{aligned}\quad (16)$$

where

$$G = C\left(\|u\|^{\frac{2(s+\alpha)(2\alpha-1)}{(2s+2\alpha+1)(1+s)-(s+\alpha)(2\alpha-1)}} \|\nabla u\|^{\frac{2(2s+2\alpha+1)(1+s)}{(2s+2\alpha+1)(1+s)-(s+\alpha)(2\alpha-1)}} + \|u\|^{\frac{s+\alpha}{3\alpha-2}} \|\nabla u\|^{\frac{2s+8\alpha-4}{3\alpha-2}} \right)$$

is an explicit polynomial. Using the Hölder inequality and the Cauchy inequality, we have

$$\int \Lambda^{1+s}(\theta e_2)\Lambda^{1+s} u dx \le \frac{1}{2}\left\|\Lambda^{1+s}\theta\right\|^2 + \frac{1}{2}\left\|\Lambda^{1+s}u\right\|^2. \quad (17)$$

Inserting (15) and (16) into (14), we obtain that

$$\frac{d}{dt}\left\|\Lambda^{1+s}u\right\|^2 + \frac{3\nu}{2}\left\|\Lambda^{1+s+\alpha}u\right\|^2 = C\left(\left\|\Lambda^{1+s}\theta\right\|^2 + \left\|\Lambda^{1+s}u\right\|^2\right) + G. \quad (18)$$

Summing (14) and (18) together, we get

$$\frac{d}{dt}\left(\left\|\Lambda^{1+s}u\right\|^2 + \left\|\Lambda^{1+s}\theta\right\|^2\right) + \nu\left\|\Lambda^{1+s+\alpha}u\right\|^2 + \kappa\left\|\Lambda^{1+s+\beta}\theta\right\|^2 = C\left(\left\|\Lambda^{1+s}\theta\right\|^2 + \left\|\Lambda^{1+s}u\right\|^2\right) + F + G. \quad (19)$$

Applying the Gronwall inequality, and by (5) and (10), we find that, for all $t \in [0,T]$

$$\left\|\Lambda^{1+s}u\right\|^2 + \left\|\Lambda^{1+s}\theta\right\|^2 + \nu\int_0^t \left\|\Lambda^{1+s+\alpha}u\right\|^2 d\tau + \kappa\int_0^t \left\|\Lambda^{1+s+\beta}\theta\right\|^2 d\tau = C,$$

where $C = C\left(\left\|\Lambda^{1+s}u_0\right\|, \left\|\Lambda^{1+s}\theta_0\right\|, \nu, \kappa, s, T\right)$ is a positive constant.

B. *The Proof of Uniqueness*

For any fixed $T > 0$, suppose there are two solution (u_1, θ_1, P_1) and (u_2, θ_2, P_2) to the Boussinesq system (1). Setting $\tilde{u} = u_1 - u_2, \tilde{\theta} = \theta_1 - \theta_1$ and $\tilde{P} = \tilde{u}_1 - \tilde{u}_2$, then $(\tilde{u}, \tilde{\theta}, \tilde{P})$ satisfies

$$\begin{cases} \tilde{u}_t + \nu\Lambda^{2\alpha}\tilde{u} + + u_1 \cdot \nabla\tilde{u} + \tilde{u} \cdot \nabla u_2 + \nabla\tilde{P} = \tilde{\theta} e_2, \\ \operatorname{div} \tilde{u} = 0, \\ \tilde{\theta}_t + \kappa\Lambda^{2\beta}\tilde{\theta} + u_1 \cdot \nabla\tilde{\theta} + \tilde{u} \cdot \nabla\theta_2 = 0, \\ \tilde{u}(x,0) = 0, \tilde{\theta}(x,0) = 0, \end{cases} \quad (20)$$

Taking the L^2 inner product of the first equation of (20) with $\tilde{u}$ and the third equation with $\tilde{\theta}$, respectively, we have

$$\frac{d}{dt}(\|\tilde{u}\|^2 + \|\tilde{\theta}\|^2) + 2\nu\|\nabla\tilde{u}\|^2 + 2\kappa\|\nabla\tilde{\theta}\|^2 \le \int \tilde{\theta} e_2 \cdot \tilde{u} dx - \int \tilde{u} \cdot \nabla u_2 \cdot \tilde{u} dx - \int \tilde{u} \cdot \nabla e_2 \tilde{\theta} dx. \quad (21)$$

Using the Hölder inequality and the Cauchy inequality, we can get as follows

$$\int_\Omega \tilde{\theta} e_2 \cdot \tilde{u} dx \le \frac{1}{2}\|\tilde{\theta}\|^2 + \frac{1}{2}\|\tilde{u}\|^2, \quad (22)$$

$$-\int_\Omega \tilde{u} \cdot \nabla u_2 \cdot \tilde{u} dx \le \left| -\int_\Omega \tilde{u} \cdot \nabla u_2 \cdot \tilde{u} dx \right| \le C\|\nabla u_2\|\|\tilde{u}\| + \frac{\upsilon}{4}\|\nabla\tilde{u}\|, \quad (23)$$

and

$$-\int_\Omega \tilde{u} \cdot \nabla\theta_2 \cdot \tilde{\theta} dx \le \left| -\int_\Omega \tilde{u} \cdot \nabla\theta_2 \cdot \tilde{\theta} dx \right| \le C\|\nabla\theta_2\|^2\|\tilde{\theta}\|^2 + \frac{\upsilon}{4}\|\nabla\tilde{\theta}\| + C\|\nabla\theta_2\|^2\|\tilde{u}\|_2 + \frac{\kappa}{2}\|\nabla\tilde{u}\|^2. \quad (24)$$

Inserting (22)-(24) into (21), we arrive at

$$\frac{d}{dt}(\|\tilde{u}\|^2 + \|\tilde{\theta}\|^2) + \upsilon\|\nabla\tilde{u}\|^2 + \kappa\|\nabla\tilde{\theta}\|^2 \le C(\|\nabla\theta_2\|^2 + \|\nabla u_2\|^2 + 1)(\|\tilde{\theta}\|^2 + \|\tilde{u}\|^2). \quad (25)$$

Using the Gronwall inequality and the estimates for θ_2 and u_2, (25) implies that, for any $t \ge 0$,

$$e^{-\mathrm{Ct}}(\|\tilde{u}\|^2 + \|\tilde{\theta}\|^2) \le (\|\tilde{\theta}(0)\|^2 + \|\tilde{u}(0)\|^2), \quad (26)$$

i.e., $\tilde{u} = 0, \tilde{\theta} = 0, \theta_1 = \theta_2, u_1 = u_2$. So the solution of the Boussinesq system (1) is unique.

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