NSD Total Choosability of Planar Graphs with Girth at Least Four

Xue Han, Jihui Wang* and Baojian Qiu
School of Mathematical Science, University of Jinan
Jinan, Shandong, P.R. China, 250022
*Corresponding author

***Abstract*—A proper total k-coloring ϕ of a graph G is a mapping from $V(G) \cup E(G)$ to $\{1, \cdots, k\}$ such that no adjacent or incident elements in $V(G) \cup E(G)$ receive the same color. Let $\Sigma_\phi(u)$ denote the sum of the colors on the edges incident with the vertex u and the color on u. A proper total k-coloring of G is called neighbor sum distinguishing if $\Sigma_\phi(u) \neq \Sigma_\phi(v)$ for each edge $uv \in E(G)$. Let L_z $(z \in V(G) \cup E(G))$ be a set of lists of integer numbers, each of size k. The smallest k for which for any specified collection of such lists, there exists a neighbor sum distinguishing total coloring using colors from L_z for each $z \in V(G) \cup E(G)$ is called the neighbor sum distinguishing total choosability of G, and denoted by $ch_\Sigma^T(G)$. In this paper, we prove that $ch_\Sigma^T(G) \leq \Delta(G) + 3$ for planar graphs with girth at least 4. This implies that Pilsniak and Wozniak' conjecture is true for any planar graphs with girth at least 4 and $\Delta(G) \geq 7$.**

Keywords-NSD total coloring; choosability; girth; planar graph

I. INTRODUCTION

The terminology and notation used but undefined in this paper can be found in [3]. Graphs considered in this paper are finite, simple and undirected. Let $G = (V, E)$ be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree and minimum degree of G, respectively. Let $d_G(v)$ or simply $d(v)$ denote the degree of a vertex v in G. A vertex v is called an l-vertex if $d(v) = l$, similarly, an l^+-vertex or an l^--vertex if $d(v) \geq l$ or $d(v) \leq l$. Let $d_i(v)$ ($d_{i^+}(v)$, $d_{i^-}(v)$) be the number of neighbors of v with degree i (at least i, at most i) in G. A k-face is a face of degree k.

Given a graph $G = (V, E)$ and a positive integer k, a total k-coloring of G is a proper coloring $\phi : V(G) \cup E(G) \to \{1, \cdots, k\}$, where a proper coloring means every pair of adjacent or incident elements receive different numbers. Given a total k-coloring ϕ of G, let $C_\phi(u)$ denote the set of colors of the edges incident to v and the color of v. A total k-coloring is called adjacent vertex distinguishing if for each edge uv, $C_\phi(u)$ is different from $C_\phi(v)$. A smallest such k is called the adjacent vertex distinguishing total chromatic number of G, denoted by $\chi_a^T(G)$. Zhang *et al.* [8] put forward the following conjecture.

Conjecture 1.1[8] For any graph G with at least two vertices, $\chi_a^T(G) \leq \Delta(G) + 3$.

Conjecture 1.1 has been proved for a few special cases, such as subcubic graphs, K_4-minor free graphs and some special planar graphs, see [2,6,7]. Recently, colorings and labellings related to sums of the colors have been studied widely, see the survey paper [1]. In a total k-coloring of G, let $\Sigma_\phi(v)$ denote the sum of colors of the edges incident to v and the color of v. If for each edge $uv \in E(G)$, we have $\Sigma_\phi(u) \neq \Sigma_\phi(v)$, we call such total k-coloring a k-neighbor sum distinguishing total coloring. The smallest number k is called the neighbor sum distinguishing total chromatic number of G, denoted by $\chi_\Sigma^T(G)$. For neighbor sum distinguishing total colorings, we give the following conjecture due to Pilsniak and Wozniak[5].

Conjecture 1.2[5] For any graph G with at least two vertices, $\chi_\Sigma^T(G) \leq \Delta(G) + 3$.

Conjecture 1.2 implies Conjecture 1.1, since it is easy to check that $\chi_a^T(G) \leq \chi_\Sigma^T(G)$. Pilsniak and Wozniak[5] proved that Conjecture 1.2 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. For a given graph G, let L_z $(z \in V(G) \cup E(G))$ be any set of list of integer numbers, each of size k. If for any specified collection of such lists, there exists a neighbor sum distinguishing total coloring of G using colors from L_z for each $z \in V(G) \cup E(G)$, we call such coloring a k-neighbor sum distinguishing list total coloring, the smallest k is called the neighbor sum distinguishing total choosability of G, and denoted by $ch_{\Sigma}^{T}(G)$. In this paper, we studied the neighbor sum distinguishing total choosability of planar graphs with girth at least 4 and proved the following result.

Theorem 1.1 If G is a planar graph with girth at least 4 and $\Delta(G) \geq 7$, then $ch_{\Sigma}^{T}(G) \leq \Delta(G) + 3$.

Clearly, $\chi_{\Sigma}^{T}(G) \leq ch_{\Sigma}^{T}(G)$, so the result above holds also for $\chi_{\Sigma}^{T}(G)$. This implies that Pilsniak and Wozniak' conjecture is true for planar graphs with girth at least 4 and $\Delta(G) \geq 7$. Our approach is based on the discharging method and some other tricks, which have been widely used in coloring theory.

II. PROOF OF THEOREM 1.1

In order to prove the main result, we need next lemma.

Lemma 2.1[4] Suppose m is a positive integer, L_j is a set of integers with $|L_j| = l_j \geq m$ for each $j \in \{1, \cdots, m\}$, let $T_m(L_1, \cdots, L_m) =$

$$\left\{ \sum_{i=1}^{m} x_i \middle| x_i \in L_i, i \neq j \Rightarrow x_i \neq x_j \right\}.$$

Then

$$|T_m(L_1, \cdots, L_m)| \geq \sum_{j=1}^{m} l_j - m^2 + 1.$$

Let L_z $(z \in V(G) \cup E(G))$ be any given set of lists of integer numbers, each of size k, where $k = \Delta(G) + 3$. For simplicity, we use "k-nsd list total coloring" to denote "k-neighbor sum distinguishing list total coloring". Let ϕ be a k-nsd list total coloring of planar graph G without adjacent triangles with $\Delta(G) \geq 7$. Assume that $u \in V(G)$ with $d(u) \leq 3$, it is easy to see that u has at most 3 adjacent vertices and 3 incident edges, and the sum obtained at u must be distinct from 3 sums at the adjacent vertices of u. So u has at most 9 forbidden colors. Since $|L_u| = k \geq 10$, we may first erase the color of u and recolor it finally. In other words, we may omit the recoloring for all 3^--vertices in the following discussion.

Our proof proceeds by reduction and absurdum. Assume that G is a counterexample to Theorem 1.1 such that $|V(G)| + |E(G)|$ is as small as possible. Obviously, G is connected. Similar to the claim in [4], we have the following claim.

Claim 1[4]. For any vertex $u \in V(G)$, it holds that $\sum_{i=1}^{3} [d_i(u)(\Delta(G) + 4 - d(u) - i)] \leq d(u) - 1$.

By Claim 1, for any $u \in V(G)$ with $d(u) \geq 4$, we have the following claim.

Claim 2. (1) There is no 4^--vertex adjacent to any 3^--vertex.

(2) If u is a 5-vertex of G, then $d_1(u) = 0$ and $d_{3^-}(u) = 0$.

(3) If u is a 6-vertex of G, then $d_{2^-}(u) \leq 1$ and if $d_2(u) = 1$, then $d_3(u) \leq 1$.

(4) If u is a l-vertex of G with $l \geq 7$, then $d_1(u) \leq \left\lfloor \frac{l-1}{3} \right\rfloor$.

Let H be the graph obtained by removing all leaves of G. By claim2, H is a connected planar graph with $\Delta(H) \geq 2$, and we have the following claim.

Claim 3. Let v be a vertex of H, if $d_H(v) = 2$ or $d_H(v) = 3$, then the neighbors of v must be 5^+-vertices in H.

In order to complete the proof, we use the discharging method. Using Euler's formula

$$|V(H)| - |E(H)| + |F(H)| = 2,$$

then $\sum_{v \in V(H)} (d_H(v) - 4) + \sum_{f \in F(H)} (d_H(f) - 4) = -8$.

First, we give an initial charge function $w(v) = d_H(v) - 4$ for every $v \in V(G)$ and $w(f) = d_H(f) - 4$ for every $f \in F(H)$. Next, we design a discharging rule and redistribute weights accordingly. Let w' be the new charge after the discharging. We will show that $w'(x) \geq 0$ for all $x \in V(H) \cup F(H)$. This leads to the following contradiction:

$$0 \leq \sum_{x \in V(H) \cup F(H)} w'(x) = \sum_{x \in V(H) \cup F(H)} w(x) = -8 < 0.$$

Hence, this demonstrates that no such a counterexample can exist. The discharging rule is defined as follow:

(R) For each 5^+-vertices u of H, gives 1 to each adjacent 2-vertex and gives $\frac{1}{3}$ to each adjacent 3-vertex.

By rule (R), we have the following results:

1. For each 5-vertex $u \in V(G)$, by claim 2, $d_H(u) = 5$ and u has at most one neighbor of 3^--vertex in H. So $w'(u) \geq 5 - 4 - 1 = 0$.

2. For each 6-vertex $u \in V(G)$, by claim 2, $d_{2^-}(u) \le 1$. And we have

(1) if $d_{2^-}(u) = 0$, then $d_H(u) = 6$, and the neighbors of u must be all 3^+-vertex in H, So $w'(u) \ge 6 - 4 - \frac{6}{3} = 0$.

(2) if $d_{2^-}(u) = 1$, then $d_3(u) \le 1$, So $w'(u) = 6 - 4 - d_{2^-}(u) - \frac{d_3(u)}{3} \ge 2 - 1 - \frac{1}{3} > 0$.

3. For each l-vertex $u \in V(G)$ with $l \ge 7$, by claim 1, we have

$$l - 1 - (\Delta - l + 3)d_1(u) - (\Delta - l + 2)d_2(u) - (\Delta - l + 1)d_3(u) \ge 0.$$

So (1) if $d_{2^-}(u) = 0$, then

$$w'(u) = l - 4 - d_{2^-}(u) - \frac{d_3(u)}{3} \ge l - 4 - \frac{l}{3} > 0.$$

(2) if $d_{2^-}(u) = 1$, then

$$w'(u) = l - 4 - d_{2^-}(u) - \frac{d_3(u)}{3}. \ge l - 4 - 1 - \frac{l-1}{3} \ge 0$$

(3) if $d_{2^-}(u) = 2$, by claim 1, we have

$$3d_1(u) + 2d_2(u) + d_3(u) \le l - 1,$$

which induces that $d_3(u) \le l - 5$. Then

$$w'(u) = l - 4 - d_{2^-}(u) - \frac{d_3(u)}{3}. \ge l - 4 - 2 - \frac{l-5}{3} > 0$$

(4) if $d_{2^-}(u) \ge 3$, then

$$\begin{aligned} w'(u) &= l - 4 - d_1(u) - d_2(u) - \frac{d_3(u)}{3} \\ &= [l - 1 - (\Delta - l + 3)d_1(u) - (\Delta - l + 2)d_2(u) - (\Delta - l + 1)d_3(u)] \\ &\quad + (\Delta - l + 2)d_1(u) + (\Delta - l + 1)d_2(u) + (\Delta - l + \frac{2}{3})d_3(u) - 3 \\ &\ge 2d_1(u) + d_2(u) + \frac{2d_3(u)}{3} \ge 0. \end{aligned}$$

4. For each 2-vertex or 3-vertex u in H, by claim 3, we have $w'(u) \ge 2 - 4 + 2 = 0$ or $w'(u) \ge 3 - 4 + 3 \cdot \frac{1}{3} = 0$.

5. For each face f in H, since H is also a planar graph with girth at least 4, then $d_H(f) \ge 4$, and we have $w'(f) = w(f) \ge 0$.

From above discussion, we have $\sum_{x \in V(H) \cup F(H)} w'(x) \ge 0$. It is a contradiction, which completes the proof of Theorem 1.1.

Acknowledgment

This work was supported by the Natural Science Foundation of Shandong Provence (ZR2016AM01) and the Scientific Research Foundation of University of Jinan (XKY1612)

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