

Weibull Distribution Based Multi-class Stochastic User Equilibrium Model

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Abstract—This paper aims to develop a multi-class Weibull distribution based route choice model (MWM) to alleviate the drawback from conventional logit model meanwhile take multi-class users into consideration. Specially, this paper distinguishes the user categories according to different cognitive levels of the network based on weibit model. The equivalent mathematical programming formulation for multi-class stochastic user equilibrium (MSUE) is provided and the solution algorithm is given. Numerical examples are also presented to illustrate the equilibrium traffic assignment.

Keywords—multi-class user; logit model; weibit model; stochastic user equilibrium; network cognitive level

I. INTRODUCTION

Researches on traffic assignment have long been focused on. The user equilibrium (UE) principle proposed by Wardrop (1952) assumes that all travelers are rational and they all have perfect knowledge of the network travel times. Daganzo and Sheffi (1977) relaxed the perfect knowledge assumption of the UE model, and suggested the stochastic user equilibrium (SUE) principle. The SUE model incorporates a random error term to capture travelers' perception error of the route travel cost. Given the distribution of the random error term, the probability that a particular alternative will be chosen by a traveler can be calculated. If the error term follows independently and identically distributed (IID) Gumbel distribution, the logit model can be derived (Dial, 1971). Logit model is widely used for it has a closed-form route choice probability expression. But logit model has two known drawbacks: (1) inability to account for the overlapping (or correlation) among routes and (2) inability to account for perception variance with respect to trips of different lengths. There are many extended logit model to handle the route overlapping problem, such as C-logit model, PSL model, PCL model, CNL model, more details are given by Chen et al. (2012). But the improvements on drawback (2) failed to get a good result.

Castillo et al. (2008) proposed Weibit route choice model, with the assumption that the error term follows Weibull distribution. The emergence of the Weibit model can effectively solve the drawback (2) of the logit model. And it also has a closed-form route choice expression. Kitthakesorn and Chen (2013, 2014) provided the equivalent MP formulations for Weibit model. Yao and Chen (2014) focused

on the stochastic assignment paradox of Logit and Weibit models.

However, the above researches of weibit model assume that travelers are homogeneous. In fact, due to the diversity of the travelers' characteristics, different categories of users should be taken into account. Yang (2004) considered the various value of time of users. Hu (2010) used a multi-user traffic model to capture different network cognitive level.

This paper proposes a Weibull distribution based multi-class route choice model with an equivalent MP formulation, distinguishes the user categories according to different cognitive levels of the network.

The remainder of this paper is organized as follows. Section 2 describes the existing route choice model. Section 3 develops the multi-class route choice model based on the weibit model. In section 4, equivalent MP formulations for MSUE and solution algorithm are provided. Numerical results are presented in section 5, and some concluding remarks are provided in section 6.

II. ROUTE CHOICE MODEL DESCRIPTION

A. Logit Model

The Logit route choice probability expression is as follows:

$$P_k^w = \frac{\exp(-\theta c_k^w)}{\sum_{l \in K^w} \exp(-\theta c_l^w)} = \frac{1}{\sum_{l \in K^w} \exp[-\theta(c_l^w - c_k^w)]} \quad (1)$$

$\forall k \in K^w, w \in W$

Where W is the set of origin-destination (O-D) pairs, K^w is the set of routes between O-D pair w , c_k^w is the travel cost of route k , and θ is the dispersion parameter. The variance of Gumbel distribution is $\pi^2 / 6\theta^2$, which is only determined by θ , cannot change according to c_k^w . As a result, logit model is not able to account for perception variance with respect to trips of different lengths.

B. Weibit Model

The Weibit route choice probability expression is as follows:

$$P_k^w = \frac{(c_k^w)^{-\beta}}{\sum_{l \in K^w} (c_l^w)^{-\beta}} = \frac{1}{\sum_{l \in K^w} (c_l / c_k)^{-\beta}} \quad \forall k \in K^w, w \in W \quad (2)$$

Where β is the shape parameter of Weibull distribution. The Weibit model does not require the homogeneous perception variance assumption, so it relaxes the identically distributed assumption of the Logit model.

The variance of Weibit distribution is

$$\text{Var}[C_k^w] = (\alpha_k^w) \left[\Gamma(1 + \frac{2}{\beta^w}) - \Gamma^2(1 + \frac{1}{\beta^w}) \right],$$

$$\alpha_k^w = \frac{c_k^w - \zeta^w}{\Gamma(1 + \frac{1}{\beta^w})}, \quad \zeta^w$$

where ζ^w is the location parameter of Weibull distribution. As a result, the variance increases with the increase of c_k^w , which is very close to the actual travel situation.

III. MULTI-CLASS WEIBIT MODEL

The demand for travel is subdivided into S classes corresponding to groups of users with different cognitive level. Assume the location parameter of Weibull distribution $\zeta^w=0$, so that the variance is only related to the shape parameter β^w . Let β_s^w denote the cognitive level for users of class s between O-D pair $s=1,2,\dots,S, w \in W$. According to section 2, the perception variance is monotonically decreasing with β_s^w , that is to say the smaller β_s^w corresponds the users' type who has better cognitive level.

According to the random utility maximization theory, people always choose the path of least perceived travel cost. Then, the probability of users of type s choosing route k , using the total probability theorem is as follows:

$$\begin{aligned} P_{sk}^w &= \Pr[C_{sk}^w = \min_{l \in K^w} (C_{sl}^w)] = \Pr[C_{sk}^w \leq \min_{l \neq k} C_{sl}^w] \\ &= \int_0^\infty \Pr[\min_{k \neq l} C_{sl}^w \geq x \mid C_{sk}^w = x] dF_{C_{sk}^w}(x) \\ &= - \int_0^\infty \prod_{l \neq k} G_{C_{sl}^w}(x) \frac{\partial C_{sk}^w(x)}{\partial x} dx \\ &\quad \forall k \in K^w, w \in W, s=1,2,\dots,S \end{aligned} \quad (3)$$

Where $F(x)$ is the cumulative distribution function, $S(x)$ is the survival function, and $S(x)=1-F(x)$.

Using the survival function of $C_{sk}^w \sim \text{Weibull}(\zeta^w, \alpha_k^w, \beta_s^w)$

$$G(t) = \exp[-(\frac{t - \zeta^w}{\alpha_k^w})^{\beta_s^w}], t \geq \zeta^w \quad (4)$$

When $\zeta^w=0$, we have the Weibull based multi-class route choice probability

$$P_{sk}^w = \frac{(c_k^w)^{-\beta_s^w}}{\sum_{l \in K^w} (c_l^w)^{-\beta_s^w}} \quad \forall k \in K^w, w \in W, s=1,2,\dots,S \quad (5)$$

When $\beta_s^w \rightarrow \infty$, $P_{sk}^w=1$, the model collapses to the UE model, travelers will choose the path with least travel cost, which declares that in the multi-class weibit model, UE users who have perfect knowledge of the network are corresponding

to the travelers with infinite β_s^w . When $\beta_s^w \rightarrow 0$, $P_{sk}^w = \frac{1}{|R^w|}$, $|R^w|$ is the number of available routes between O-D pair w . Then, traffic demand is assigned uniformly to the route, which illustrates that $\beta_s^w=0$ is corresponding to the users with little knowledge of network or not caring about the travel cost at all.

IV. EQUIVALENT MATHEMATICAL PROGRAMMING FORMULATIONS AND SOLUTION ALGORITHM

A. Equivalent Mathematical Programming Formulations

This section presents the equivalent mathematical programming (MP) formulations for the multi-class weibit route choice model under congested network. To begin with, a general assumption of link travel cost is made, i.e.,

Assumption 1. The link travel cost τ_a is a strictly increasing function w.r.t. its own flow.

Assumption 2. ζ^w is equal to zero. This assumption indicates that each route is assumed to have the same coefficient of variation.

Assumption 3. The route travel cost is a function of multiplicative link travel cost, i.e.,

$$c_k^w = \prod_{a \in K^w} \tau_a \quad (6)$$

Consider the following MP formulation:

$$\min Z = \sum_{a \in A} \int_0^{v_a} \ln \tau_a(\omega) d\omega + \sum_{s=1}^S \sum_{w \in W} \frac{1}{\beta_s^w} \sum_{k \in K^w} f_{sk}^w (\ln f_{sk}^w - 1) \quad (7)$$

s.t.

$$\sum_{k \in K^w} f_{sk}^w = q_s^w, \forall w \in W, s = 1, 2, \dots, S \quad (8)$$

$$f_{sk}^w \geq 0, \forall k \in K^w, w \in W, s = 1, 2, \dots, S \quad (9)$$

$$v_a = \sum_{s=1}^S \sum_{w \in W} \sum_{k \in K^w} f_{sk}^w \delta_{a,k}^w, \forall k \in K^w, w \in W, a \in A \quad (10)$$

Where f_{sk}^w is the flow on route k of the s type users between O-D pair w , q_s^w is the travel demand of the s type users between O-D pair w , v_a is the link flow on link a . Eqs. (8) and (9) are respectively the flow conservation constraint and the non-negativity constraint. Eqs. (10) is the flow relationship between link and path.

Proposition 1. The MP formulation given in Eqs.(7)-(10) has the solution of the multi-class weibit model.

Proof. The KKT conditions of (7) -(10) are:

$$\sum_{a \in A} \ln \tau_a \delta_{a,k}^w + \frac{1}{\beta_s^w} \ln f_{sk}^w - \lambda_s^w - \mu_{sk}^w = 0 \quad (11)$$

$$\mu_{sk}^w f_{sk}^w = 0 \quad (12)$$

$$\mu_{sk}^w \geq 0 \quad (13)$$

Eqs. (8) and (9)

Where λ_s^w, μ_{sk}^w are the dual variable associated with Eqs.(8) and (9). When $f_{sk}^w > 0$, the dual variable

$$\mu_{sr}^{ij} = \sum_{a \in A} \ln \tau_a \delta_{a,r}^{ij} + \frac{1}{\beta_s^{ij}} \ln f_{sr}^{ij} - \lambda_s^{ij} = 0, \text{ i.e.,}$$

$$\lambda_s^w = \sum_{a \in A} \ln \tau_a \delta_{a,k}^w + \frac{1}{\beta_s^w} \ln f_{sk}^w$$

From assumption 3,

$$\sum_{a \in A} \ln \tau_a \delta_{a,k}^w = \ln c_k^w, \lambda_s^k = \ln c_k^k + \frac{1}{\beta_s^w} \ln f_{sk}^w$$

as a result

$$f_{sk}^w = \exp(\beta_s^w \lambda_s^w) \cdot (c_k^w)^{-\beta_s^w} \quad (14)$$

From Eqs.(8),
leads to the multi-class route choice probability as:

$$p_{sk}^w = \frac{(c_k^w)^{-\beta_s^w}}{\sum_{l \in K^w} (c_l^w)^{-\beta_s^w}} \quad (15)$$

Thus, the MP formulation given in Eqs. (7) -(10) corresponds to the multi-class SUE model for which the route-flow solution is obtained according to the multi-class Weibit model. This completes the proof.

B. Solution Algorithm

From assumption 3, since the link cost is in a multiplicative form, this paper use the logarithm operator to transform it to an additive form, i.e.,

$$\bar{\tau}_a = \ln \tau_a \quad (16)$$

This process facilitates the use of an ordinary shortest path algorithm. The route cost can then be determined from

$$c_k^w = \exp(\sum_{a \in A} \bar{\tau}_a \delta_{a,k}^w), \forall k \in K^w, w \in W \quad (17)$$

The solution algorithm is as follows:

Step 1. Set the initialization flow f meet the travel demand;

Step 2. Update link travel cost $\bar{\tau}_a$ and route travel cost c_k^w ;

Step 3. Direct finding. Yields flows $y = P(f)$ (Where $P(\cdot)$ is the probability of multi-class Weibit model);

Step 4. Line search. Find $\alpha \in [0,1]$ solves $\alpha = \arg \min Z(f + \alpha(y - f))$;

Step 5. Move. Set $x = f + \alpha(y - f)$;

Step 6. Convergence test. If $|x - f| < \varepsilon$, stop; otherwise set $|x - f| < \varepsilon$, and go to step 2.

It should be noted that $\bar{\tau}_a$ could be negative if τ_a is less than one. When the network contains a negative cost loop, all negative $\bar{\tau}_a$ would be set to a very small positive number, and the algorithm is then repeated.

V. NUMERICAL EXAMPLES

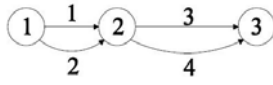


FIGURE I. A SIMPLE NETWORK FOR NUMERICAL ANALYSIS

This example network shown in fig.1 consists of 3 nodes, 4 links, and 4 alternative paths between O-D pair (1-3). For convenience, we consider only two types of travelers in this example, i.e., low cognitive level and high cognitive level. The travel demand of every type of travelers is 50vph. The link-path incident matrix and the link travel cost functions are as follows:

$$\Delta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{aligned} t_1 &= 3 + f / 20 \\ t_2 &= 6 + f / 20 \\ t_3 &= 103 + f / 20 \\ t_4 &= 106 + f / 20 \end{aligned}$$

To make a comparison, table 1 shows the equilibrium flow results of MLM and MWM. The parameters of logit model are set to $\theta_1 = 0.49$ for low cognitive level and $\theta_2 = 0.71$ for high cognitive level. Correspondingly, the parameters of weibit model are set to $\beta_1 = 2$ and $\beta_1 = 3$.

TABLE I. EQUILIBRIUM FLOW RESULTS OF MLM AND MWM

Flow results		Multi-class logit model		Multi-class weibit model	
		$\theta_1 = 0.49$	$\theta_2 = 0.71$	$\beta_1 = 2$	$\beta_1 = 3$
Path flow	f_1	20.96	24.81	15.74	17.32
	f_2	11.41	10.41	14.92	15.98
	f_3	11.41	10.41	9.93	8.68
	f_4	6.22	4.37	9.41	8.02
Link flow difference	$v_1 - v_2$	14.74	20.44	11.32	16.60
	$v_3 - v_4$	14.74	20.44	1.34	2.00

The result shows:

(1) From the equilibrium path flows, the MLM path flows of path 2 and path 3 are equal, because there is no absolute travel cost difference between path 2 and path 3 (i.e., $(3 + f / 20 + 106 + f / 20) - (6 + f / 20 + 103 + f / 20) = 0$); but the MWM path flows of path 2 and path 3 are different, because the relative travel cost difference is not equal $(3 + f / 20 + 106 + f / 20) / (6 + f / 20 + 103 + f / 20)$ (i.e., $= (1090 + f) / (590 + f) \neq 1$).

(2) From the equilibrium link flow difference, $v_1 - v_2$ and $v_3 - v_4$ of MLM are equal, illustrates that logit model can only capture the absolute travel cost difference. As shown in Figure II (A), the relationship between travel cost and utility is linear,

i.e., the reduction rate of utility does not change respect to the increase of travel cost. But for MWM, $v_1 - v_2$ is larger than $v_3 - v_4$, declares that even there exists absolute travel cost difference, with the increase of travel cost, people's awareness to absolute difference shows diminishing marginal (i.e., people feel obvious difference between 3 and 6, but the difference shrinks between 103 and 106). As shown in Figure II (B), the relationship between travel cost and utility is nonlinear, illustrates the marginal effect in the perception utility.

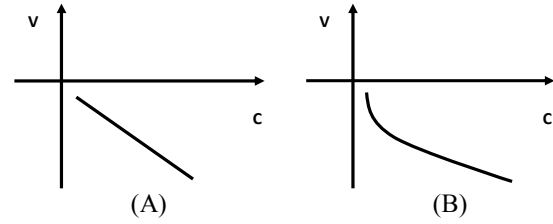


FIGURE II. UTILITY FUNCTION OF LOGIT MODEL(A)AND WEIBIT MODEL(B)

(3) From the various types of users, no matter MLM or MWM, path one's flow (the shortest path) of the first type users is always smaller than the second type users. That is to say the second type users have more perfect knowledge of the network, i.e., the probability of least route choice of users with high cognitive level is high. By setting different model parameters, various users with different cognitive level can be distinguished, which is more realistic.

VI. CONCLUSIONS

This paper developed a closed-form multi-class weibit route choice model. Equivalent MP formulation for MSUE is provided and the solution algorithm is given. The perception error term in the utility function follows Weibull distribution can make the perception variance related to the travel cost, and then the marginal effect in the process of route choice can be captured. Meanwhile, the exist weibit model is modified to distinguish the user categories according to the different cognitive level of the network.

Further research should focus on the drawback of weibit model, more general route choice model should be developed. Other heterogeneity of users and traveler behavioral factors should be taken into account, in order to better explain the travel choice behavior.

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