Investigation on 1-D and 2-D Signal Sparsity Using the Gini Index, L1-Norm and L2-Norm for the Best Sparsity Basis Selection

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Abstract. Sparsity of signals is a crucial fundamental concept in diverse fields such as compressed sensing, image processing, dictionary learning, blind source separation and sampling theory. The objective of this paper is to present sparsity analysis of 1-D speech signal and 2-D image using Gini index, L1-norm and L2-norm, for the best sparsity basis selection. The DWT families, FFT, DCT, LPC and PCA are used as sparsifying basis. The result shows that the dmey wavelet (1-level decomposition) and bior3.7 wavelet (3-level decomposition) show the greatest value of Gini index for speech. Furthermore, the bior5.5 wavelet shows the lowest value of L1-norm and L2-norm. The DCT exhibits largest Gini index compared to FFT, LPC and PCA for speech. For image signals, the bior3.7 (1-level decomposition) and bior3.1 (3-level decomposition) exhibits highest Gini index. Moreover, the bior3.1 and the bior5.5 wavelet show the lowest value of L1-norm and L2-norm. The PCA exhibits the highest Gini index for image.

Keywords: Sparsity, Gini Index, Lorenz Curve, Norms, FFT, DWT, DCT, PCA, Compressed Sensing.

1 Introduction

Sparsity is defined as the number of non-zero coefficients in the signal under some transform domain, for a better representation of a signal. The sparsity is a key concept applied in different signal processing areas such as compressed sensing [1, 2, 3], image processing [4, 5], dictionary learning [6], medical imaging [7], sampling theory [8, 9], blind source separation [10, 11], etc.

There are different methods used for measuring sparsity of a signal. Some of the popular sparsity measures are: L^0, L^0_ε, L^1, L^p, $\frac{L^2}{L^1}$, $tanh_{a,b}$, log, kurtosis, Hoyer, pq-mean and Gini index [12, 13]. These methods calculate the values which will represent the sparsity of a signal vector $\vec{u} = [u_1, u_2, \ldots, u_N]$.

Abbreviations	
DFT-Discrete Fourier Transform	PCA-Principal Component Analysis
DCT- Discrete Cosine Transform	LPC-Linear Prediction Coding
DWT- Discrete Wavelet Transform	CS – Compressed Sensing

The L^p-norm is widely studied and applied sparsity measure. The L0-norm is the traditional sparsity metric which counts the number of non-zero elements in the signal vector given by: $\|\vec{u}\| = u_j \neq 0,\ j = 1,2,\ldots,N$. However, it is practically unsuitable because of two disadvantages. The first drawback is the zero derivative of L^p-norm and thus exhibits zero information. Secondly, it has a very poor performance in the presence of noise. The only way to find the sparsest solution is the exhaustive search method, which is practically NP-hard and thus not used in optimization problems. Hence, L1-norm is most popular sparsity metric. In [12, 13] presented six different attributes of sparsity measures namely: Scaling, Robin Hood, Rising Tide, Bill Gates, Cloning and Babies, and the only measures that are satisfying all of these criteria are pq-means and Gini index.

In many signal processing applications, it is commonly desired to recover the discrete signal with only few numbers of measurements. For example, the success of compressed sensing (CS) [14, 15] applications is mainly depends on the sparse or compressible nature of the signal, allowing the efficient acquisition, compression and reconstruction of a signal.

The major contributions of the proposed work are: (1) Presented and investigated different signal sparsity measures such as Gini index (GI), L1-norm and L2-norm. (2) Demonstrate the 1-D speech signal sparsity

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analysis with different sparsifying transforms such as Discrete Wavelet Transform (DWT) family, FFT, DCT, LPC and PCA, using sparsity metrics like Gini index (GI), L1-norm and L2-norm. Here, we have investigated for the best sparsity transform for speech signals. (3) Demonstrate the 2-D image signal sparsity analysis with different sparsifying transforms such as Discrete Wavelet Transform (DWT) family, DCT and PCA, with sparsity metrics like Gini index (GI), L1-norm and L2-norm. Here, also we have investigated for the best sparsity transform for 2-D image signals.

The paper is organized as follows: **Section 2** introduces the different signal sparsity measurement metrics. **Section 3** presents the different sparsifying transform. **Section 4** illustrates the experimental results and discussions. Finally, **Section 5** presents the conclusions.

2 Different Signal Sparsity Metrics

There are different metrics used for the measurement of signal sparsity. The following are some important and commonly used sparsity measures.

A. Gini Index (GI)

With a vector $\vec{u} = [u_1, u_2, \ldots, u_N]$, it is ordered from smallest to largest, $u_{(1)} \leq u_{(2)} \leq \cdots \leq u_{(N)}$. Where (1), (2)… (*N*) are the newest indices after the sorting operation. The Gini index is given by equation (1) as follows:

$$S(\vec{u}) = 1 - 2\sum_{j=1}^{N} \frac{u_{(j)}}{\|\vec{u}\|_1}\left(\frac{N - j + \frac{1}{2}}{N}\right) \quad (1)$$

The Gini index [16, 17, 18, 19, 20, 21] is defined as double the area between the Lorenz curve and the equality diagonal. The Fig. 1 shows the graphical interpretation of the Gini index (Double the yellow area). The Gini index can have a value anywhere from 0 to 1. The perfect equity is indicated by zero Gini index and 1 shows perfect inequity. The larger ratio indicates the more inequitable distribution of the signal sparsity. Hence, a large value of Gini index indicates more signal sparsity. The Gini index shows the best property, where the 45 degrees line (in red) indicates the minimum sparse distribution, i.e. almost all the entries being equal.

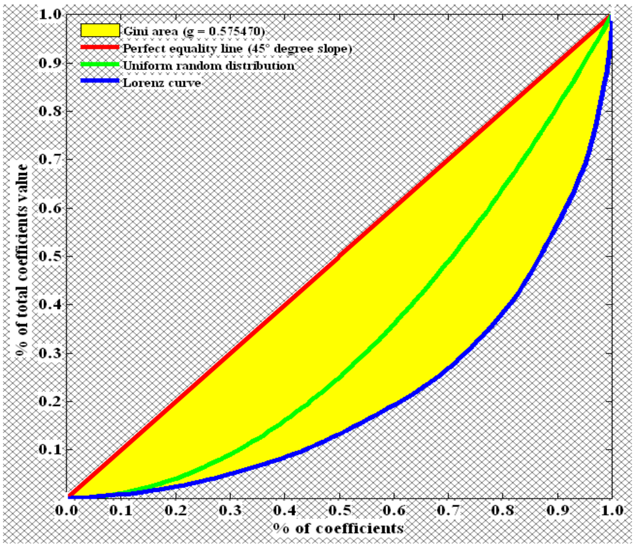


Fig. 1. Graphical interpretation of the Gini index (Double the yellow area)

B. L1-norm

The L1-norm is calculated by summing the absolute values of all the coefficients of a signal and is given by:

$$\|u\|_1 = \sum_j u_j \tag{2}$$

The smallest value of addition indicates the maximum signal sparsity. The L1-norm is a more convex function compared to other norms, and hence commonly used in solving optimization problems. The **Fig. 2** shows the diamond shaped geometrical interpretation of L1-norm. The point at which the red line (solution plane) touches to the diamond is called as exact sparse solution. Hence, it is widely used in the optimization to find out the exact sparse solution.

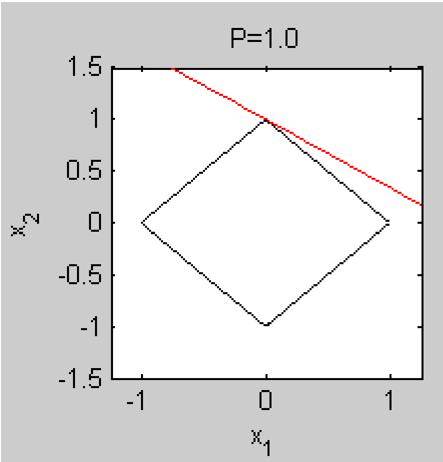


Fig. 2. Geometrical interpretation of L1-norm

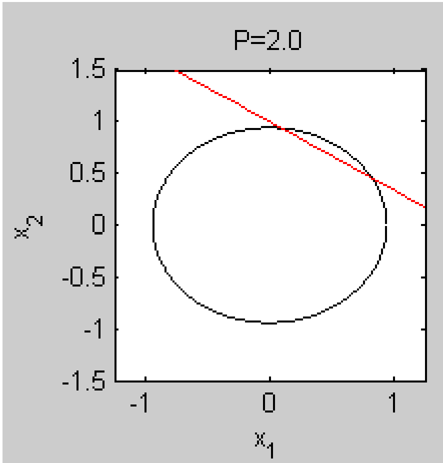


Fig. 3. Geometrical interpretation of L2-norm

C. L2-norm

The L2-norm will not provide the exact sparse solution as shown by the geometrical interpretation of L2-norm in **Fig. 3** and is calculated as follow:

$$\|u\|_2 = \left(\sum_j u_j^2\right)^{\frac{1}{2}} \tag{3}$$

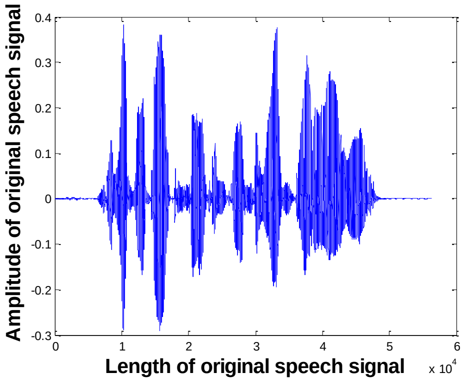
3 Scarifying Transform

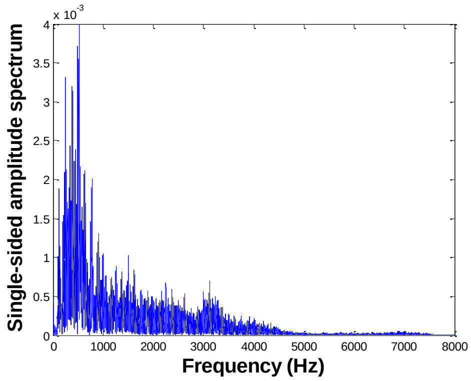
There are different scarifying transform used for the representation of signal. Some important transforms are discussed as below.

A. Discrete Fourier Transform (DFT)

The DFT is the basic sparsity transform for a signal of length *n*. But, DFT suffers from a drawback of huge number of additions and multiplication operations, resulting in increased computational cost. The Fast Fourier Transform (FFT) algorithm is used instead of DFT because of the low computational complexity of FFT i.e. $O(n \log n)$ instead of $O(n^2)$. However, the FFT have drawbacks like applying CS to a complex valued signal is a complex and weariness process. Also, FFT calculates both positive as well as negative side frequencies [22]. For input signal *x* the DFT is given by:

$$DFT\ Y(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \tag{4}$$





The **Fig. 4** shows the original speech signal of length (N= 56273) and **Fig. 5** shows the DFT of speech signal.

B. Discrete Cosine Transform (DCT)

The drawbacks of the FFT can be overcome using the DCT which has excellent energy compaction for highly correlated data. The DCT eliminate the redundancy among the neighboring values, and thus generate transform coefficients which are uncorrelated and encoded independently. Unlike the FFT, the DCT packs most of the signal energy into a few numbers of coefficients. Also, the DCT is a member of a family of real-valued discrete sinusoidal unitary transformation [23]. One of the measures of sparsity is Gini index and DCT shows high Gini index (GI=0.7387). The **Fig. 6** shows the DCT of speech signal of length (N=56273). It has very few numbers of non-zero coefficients. The speech signal sparsity can be increased by thresholding the non-significant coefficients. The DCT of signal *x* is given as:

$$DCT(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos(\pi(2n-1)(k-1)/2N) \tag{5}$$

Where,

$$\alpha(k) = \begin{cases} 1/\sqrt{N} & for\ k = 1 \\ \sqrt{2/N} & for\ \ 2 \le k \le N \end{cases} \tag{6}$$

C. Wavelet Transform

The wavelet transform (WT) [24, 25, 26] gives both the time resolution and frequency resolution of the signal. The wavelet transform is used to overcome the drawbacks of the short-time Fourier transform (STFT), i.e. for the analysis of the non-stationary signals. The STFT shows a constant resolution, while the wavelet transform shows a multi-resolution with respect to frequencies. The wavelet transform gives the approximations of the signal with big wavelets, while small details are given by small scales. The basic idea of the wavelet transform is to represent the signal to be analyzed as a superposition of wavelets. The wavelet transform is a most popular signal analysis tool used in different application areas such as speech or audio and image compression. There are different discrete wavelet transform (DWT) families available for representation of the signal like: Haar, Daubechies, Biorthogonal, Reverse Biorthogonal, Coiflets, Symlets, etc. the **Fig. 7** shows the DWT of speech signal of length (N=56273) with approximation and details.

D. Linear Prediction Coding (LPC)

The Linear predictive coding (LPC) [27] is a fundamental and most powerful speech analysis tool used for representation of the speech spectral envelope in the compressed form. The LPC not only provides the sparse coefficients, but also it gives formants, pitch, an estimate of the intensity and the frequency of the buzz, and the residual signal, which contains most of the speech information. The LPC has applications in speech compression, modeling, representation, etc [28, 29]. The **Fig. 8** shows the LPC of speech signal of length (N=56273).

E. Principal Component Analysis (PCA)

The Principal component analysis [30, 31] is a widely used dimensionality reduction technique having large sets of data e.g. microarray analysis. The main objective of PCA is to reduce the feature space dimensionality while keeping data variance. In PCA, the first covariance matrix of the signal is computed, and then Eigen values and the Eigen vectors of the covariance matrix are calculated. The Eigen vectors corresponding to the largest Eigen values are selected and used for projecting feature space to low dimension space. Thus, the large amount of signal energy is packed in few principal components of PCA which are orthogonal to each other's and thus provides the required signal sparsity [32]. The **Fig. 9** shows the PCA of speech signal of length (N=56273).

4 Experimental Results and Discussions

- **Research Methodology**

The experimentation is performed on the two test signals namely: 1-D speech signal and 2-D image. The test speech signal is available on the CMU/CSTR KDT US English TIMIT database for speech synthesis by Carnegie Mellon University and Edinburgh University [33]. The details of speech file are as follows: File name: Kdt_001.wav, Bit rate: 256 kbps, Audio sample rate: 16 kHz, Total Duration: 3 seconds. The second test signal is 210 x 210 Brain MRI image. The simulation results are generated on MATLAB 7.8.0 (R2009A) with INTEL (R) CORE 2 DUO CPU, 3 GB RAM.

4.1 1-D Speech Signal Sparsity Analysis using Gini Index, L1-norm and L2-norm

In this section we have presented the analysis of speech signal sparsity with different sparsity transforms such as Discrete cosine transform(DCT), Fast Fourier transform (FFT), Linear Prediction coding (LPC), Principal component analysis (PCA) and discrete wavelet transform(DWT) families like: Haar, Daubechies, Biorthogonal, Reverse Biorthogonal, Coiflets, Symlets, etc. The sparsity analysis is performed using different sparsity measures such as Gini Index, L1-norm and L2-norm. Furthermore, the greater value of Gini index indicates the greater sparsity of the speech signal, whereas for L1-norm lowest the value, greater is the sparsity of a signal. Also, for L2 -norm smaller the value, sparser is the signal.

The **Table 1** shows, the speech signal sparsity analysis of the Discrete Wavelet Transform (DWT) family using Gini Index, L1-norm and L2-norm. The result shows that, the bior3.7 wavelet (3-level decomposition) exhibits the greatest value of the Gini index (GI=0.8737). The bior3.5 and the bior3.9 wavelet with GI=0.8732 could be the second best choice. The bior5.5 wavelet shows the lowest value of the L1-norm (592.6113) and L2-norm (7.7235). The rbio2.8 wavelet with L1-norm= 656.0824 and the Rbio3.9 with L2-norm=8.1784 will be the second alternatives.

For the DWT with 1-level decomposition, the **Table 1** shows that the dmey wavelet exhibits the greatest value of Gini index (i.e. GI=0.8208). The bior3.9 wavelet with GI=0.8201 and the bior3.7 with GI= 0.8200 could be the second best choices as there is a very small difference in their values. From the **Table 1** it is also seen that the bior5.5 wavelet shows the lowest value of the L1-norm (873.4284) and L2-norm (9.3710). The dmey wavelet with L1-norm=882.3817 and the Rbio3.9 with L2-norm=9.4742 could be the second alternatives for speech signal sparsity.

The result shows that the sparsity level of signal increases as there is an increase in the decomposition level of wavelet family. Hence, the DWT family with 3-level decomposition results in increase in the sparsity level of signal compared to the DWT family with 1-level decomposition.

Table 1. A comparative analysis of speech signal sparsity for Discrete Wavelet Transform (DWT) family using the Gini index, L1-norm and L2-norm

Sr. no.	Sparsifying Transform	Sparsity measures with *3-level decomposition*			Sparsity measures with *1-level decomposition*		
		Gini index (GI)	L1-norm	L2-norm	Gini index (GI)	L1-norm	L2-norm
1.	db1	0.8008	809.6920	9.7816	0.7713	972.8646	9.7816
2.	db2	0.8273	737.3517	9.7816	0.7907	932.2936	9.7816
3.	db3	0.8362	710.4047	9.7816	0.7983	917.5741	9.7816
4.	db4	0.8410	705.2055	9.7816	0.8024	909.8447	9.7816
5.	db5	0.8437	696.4810	9.7816	0.8050	905.1808	9.7816
6.	db6	0.8461	694.9384	9.7816	0.8072	901.8289	9.7816
7.	db7	0.8479	689.9088	9.7816	0.8089	899.5069	9.7816
8.	db8	0.8492	688.8183	9.7816	0.8101	897.7361	9.7816
9.	db9	0.8508	685.0001	9.7816	0.8113	895.6545	9.7816
10.	db10	0.8521	683.6133	9.7816	0.8127	893.4349	9.7816
11.	sym2	0.8273	737.3517	9.7816	0.7907	932.2936	9.7816
12.	sym3	0.8362	710.4047	9.7816	0.7983	917.5741	9.7816
13.	sym4	0.8397	703.3294	9.7816	0.8018	912.0829	9.7816
14.	sym5	0.8431	703.1120	9.7816	0.8043	907.2436	9.7816
15.	sym6	0.8453	695.4730	9.7816	0.8067	903.6103	9.7816
16.	sym7	0.8481	695.9099	9.7816	0.8088	899.0634	9.7816
17.	sym8	0.8495	689.1862	9.7816	0.8099	898.2663	9.7816
18.	bior1.1	0.8008	809.6920	9.7816	0.7713	972.8646	9.7816
19.	bior1.3	0.8077	868.0260	10.7937	0.7723	986.2193	9.9701
20.	bior1.5	0.8093	883.0841	11.1084	0.7725	989.9144	10.0183
21.	bior2.2	0.8514	761.7358	11.3412	0.8052	925.4935	10.0261
22.	bior2.4	0.8527	780.7527	11.6811	0.8054	929.6822	10.0861
23.	bior2.6	0.8528	792.3741	11.8326	0.8055	932.0107	10.1184
24.	bior2.8	0.8532	801.8361	11.9295	0.8056	933.5522	10.1396
25.	bior3.1	0.8712	858.0835	14.1585	0.8192	921.7847	10.2518
26.	bior3.3	0.8731	847.3781	13.7606	0.8196	926.0613	10.3253
27.	bior3.5	0.8732	849.4339	13.6860	0.8198	928.6592	10.3676
28.	bior3.7	**0.8737**	851.2708	13.7478	0.8200	930.4188	10.3963
29.	bior3.9	0.8732	858.0313	13.8912	0.8201	931.7157	10.4175
30.	bior4.4	0.8440	661.8385	9.2328	0.8070	894.2663	9.6635
31.	bior5.5	0.8334	**592.6113**	**7.7235**	0.8056	**873.4284**	**9.3710**
32.	bior6.8	0.8530	692.7291	9.9413	0.8120	898.5766	9.8258
33.	rbio 1.1	0.8008	809.6920	9.7816	0.7713	972.8646	9.7816
34.	rbio1.3	0.8321	682.6080	9.1578	0.7975	910.6444	9.6648
35.	rbio1.5	0.8402	662.1787	9.0993	0.8048	896.7190	9.6476
36.	rbio2.2	0.8017	785.1192	9.2516	0.7731	962.5368	9.6595
37.	rbio2.4	0.8168	695.3644	8.6318	0.7878	923.2466	9.5727
38.	rbio2.6	0.8245	668.6529	8.4854	0.7948	907.3536	9.5451
39.	rbio2.8	0.8293	656.0824	8.4273	0.7994	897.9255	9.5311
40.	rbio3.1	0.7618	1325.5	13.3158	0.7312	1171	10.5086
41.	rbio3.3	0.7905	845.4423	9.3851	0.7638	990.4825	9.6671
42.	rbio3.5	0.8006	748.0424	8.6295	0.7753	951.5696	9.5507
43.	bio3.7	0.8076	710.6927	8.3486	0.7823	931.5568	9.5015
44.	bio3.9	0.8125	688.2152	8.1784	0.7875	918.4846	9.4742
45.	rbio4.4	0.8352	768.2654	10.4786	0.7954	934.6663	9.9147
46.	rbio5.5	0.8519	855.4873	12.7699	0.8028	948.2945	10.2891
47.	rbio6.8	0.8419	704.3037	9.6705	0.8044	904.8827	9.7429
48.	coif1	0.8279	732.9013	9.7816	0.7913	931.3377	9.7816
49.	coif2	0.8410	705.4458	9.7816	0.8025	910.9446	9.7816
50.	coif3	0.8470	690.7668	9.7816	0.8075	902.4424	9.7816
51.	coif4	0.8506	690.4547	9.7816	0.8108	897.2527	9.7816
52.	coif5	0.8532	680.7731	9.7816	0.8131	893.6535	9.7816
53.	dmey	0.8621	669.1432	9.7816	**0.8208**	882.3817	9.7816

Table 2. A comparative analysis of speech signal sparsity for DCT, FFT, LPC and PCA using the Gini Index, L1-norm and L2-norm

Sr. no.	Sparsifying Transform	Gini index (GI)	L1-norm	L2-norm
1.	DCT	**0.7387**	**1001.7**	**9.7816**
2.	FFT	0.7026	263560	2320.4
3.	LPC	0.5002	13390	110.5893
4.	PCA	0.6954	1200	**9.7816**

The speech signal sparsity analysis for the DCT, FFT, LPC and PCA using Gini Index, L1-norm and L2-norm is shown by the **Table 2**. From the **Table 2** it is observed that the DCT exhibits the largest Gini index (GI=0.7387). The FFT with GI=0.7026 will be the second choice. It is also seen from **Table 2** that the lowest value of the L1-norm (1001.7) and L2-norm (9.7816) are shown by the DCT. The PCA could be the second choice with the lowest value of L1-norm (1200) and the L2-norm (9.7816).

4.2 2-D Image Signal Sparsity Analysis using Gini Index, L1-norm and L2-norm

This section presents the 2-D image (as shown in **Fig. 10**) sparsity analysis for the different sparsity transforms such as Discrete cosine transform (DCT), Fast Fourier transform(FFT), Linear Prediction coding (LPC), Principal component analysis (PCA) and discrete wavelet transform(DWT) families like: Haar, Daubechies, Biorthogonal, Reverse Biorthogonal, Coiflets, Symlets, etc.

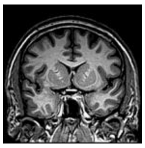


Fig. 10. 2-D test image signal (Example: MRI Coronal Brain.JPG)

Table 3. A comparative analysis of 2-D image sparsity for Discrete Wavelet Transform (DWT) family using the Gini Index, L1-norm and L2-norm (Example Brain MRI)

Sr. no.	Sparsifying Transform	Sparsity measures with *3-level decomposition*			Sparsity measures with *1-level decomposition*		
		Gini index (GI)	L1-Norm	L2-norm	Gini index (GI)	L1-norm	L2-norm
1.	db1	0.9186	1848500	19945	0.8165	2398000	19945
2.	db2	0.9211	1824100	19945	0.8211	2359200	19945
3.	db3	0.9231	1801100	19945	0.8218	2346100	19945
4.	db4	0.9227	1805100	19945	0.8219	2346300	19945
5.	db5	0.9230	1806100	19945	0.8222	2346700	19945
6.	db6	0.9228	1807600	19945	0.8229	2343100	19945
7.	db7	0.9227	1809100	19945	0.8236	2337600	19945
8.	db8	0.9225	1813700	19945	0.8238	2338100	19945
9.	db9	0.9221	1820100	19945	0.8236	2341800	19945
10.	db10	0.9218	1828200	19945	0.8232	2345900	19945
11.	sym2	0.9211	1824100	19945	0.8211	2359200	19945
12.	sym3	0.9231	1801100	19945	0.8218	2346100	19945
13.	sym4	0.9240	1787800	19945	0.8233	2334200	19945
14.	sym5	0.9224	1802500	19945	0.8230	2338500	19945
15.	sym6	0.9244	1783600	19945	0.8240	2329000	19945
16.	sym7	0.9225	1802500	19945	0.8222	2342500	19945
17.	sym8	0.9241	1785200	19945	0.8242	2327500	19945
18.	bior1.1	0.9186	1848500	19945	0.8165	2398000	19945
19.	bior1.3	0.9168	1912200	20425	0.8190	2407300	20119
20.	bior1.5	0.9152	1946100	20613	0.8198	2413100	20178
21.	bior2.2	0.9299	1797500	20904	0.8293	2316300	20203
22.	bior2.4	0.9288	1823900	21030	0.8299	2327100	20277
23.	bior2.6	0.9272	1845700	21094	0.8301	2333400	20315

24.	bior2.8	0.9263	1859400	21138	0.8303	2337600	20339
25.	bior3.1	**0.9355**	1969800	24035	0.8340	**2292600**	20440
26.	bior3.3	0.9319	1923900	22904	0.8344	2310300	20549
27.	bior3.5	0.9301	1934500	22771	0.8346	2319500	20601
28.	bior3.7	0.9302	1941600	22813	**0.8347**	2325700	20633
29.	bior3.9	0.9295	1955000	22859	0.8347	2330300	20656
30.	bior4.4	0.9256	1745100	19746	0.8232	2312400	19841
31.	bior5.5	0.9221	**1733900**	**19102**	0.8160	2338600	**19570**
32.	bior6.8	0.9255	1772000	20039	0.8253	2315700	19973
33.	rbio 1.1	0.9186	1848500	19945	0.8165	2398000	19945
34.	rbio1.3	0.9246	1754400	19688	0.8217	2330900	19841
35.	rbio1.5	0.9245	1750600	19662	0.8224	2320700	19828
36.	rbio2.2	0.9062	1979500	19869	0.8073	2464800	19867
37.	rbio2.4	0.9145	1864700	19565	0.8128	2401000	19756
38.	rbio2.6	0.9163	1837500	19491	0.8142	2384000	19727
39.	rbio2.8	0.9169	1829600	19466	0.8149	2376600	19715
40.	rbio3.1	0.8529	2674500	22580	0.7712	2805000	20884
41.	rbio3.3	0.8903	2168000	20415	0.7959	2548300	19973
42.	rbio3.5	0.8992	2.050300	19964	0.8016	2492400	19816
43.	bio3.7	0.9027	2003700	19772	0.8039	2470300	19760
44.	bio3.9	0.9050	1975400	19667	0.8050	2459000	19731
45.	rbio4.4	0.9253	1905900	21368	0.8220	2374800	20071
46.	rbio5.5	0.9208	1823200	19912	0.8295	2370300	20480
47.	rbio6.8	0.9216	1821400	19945	0.8219	2348100	19925
48.	coif1	0.9234	1793600	19945	0.8206	2363500	19945
49.	coif2	0.9238	1789400	19945	0.8231	2336200	19945
50.	coif3	0.9238	1789600	19945	0.8237	2330300	19945
51.	coif4	0.9238	1789600	19945	0.8240	2328200	19945
52.	coif5	0.9238	1791300	19945	0.8241	2327300	19945
53.	dmey	0.9237	1803000	19945	0.8246	2328400	19945

The sparsity analysis is performed with different sparsity measures such as Gini Index, L1-norm and L2-norm. The **Table 3** shows the signal sparsity analysis of Discrete Wavelet Transform (DWT) family for 2-D Image (Example. Brain MRI).

The **Table 3** shows that the bior3.7 (1-level decomposition) exhibits highest Gini index (GI=0.8347). It is also observed from the **Table 3** that the bior3.1 exhibits the lowest L1-norm (2292600) and the bior5.5 shows the minimum L2- norm (19570).

For the DWT with 3-level decomposition, the bior3.1 wavelet attains the highest Gini index (GI=0.9355). Further, the bior5.5 achieves the lowest L1-norm (1733900) along with the lowest L2-norm (19102).

The **Table 4** shows the 2-D image sparsity analysis for the DCT, FFT, LPC and PCA using Gini Index, L1-norm andL2-norm.

Table 4. A comparative analysis of 2-D image sparsity for DCT, FFT, LPC and PCA using Gini index, L1-norm and L2-norm (Example Brain MRI)

Sr. no.	Sparsifying Transform	Gini index (GI)	L1-norm	L2-norm
1.	DCT	0.9513	10439	18809
2.	DWT(1-Level)	0.8347 (bior3.7)	2292600 (bior3.1)	19570 (bior5.5)
3.	PCA	**0.9533**	**7079.6**	**7775.0**

The **Table 4** shows that the PCA exhibits the highest Gini index (GI=0.9533). It is also observed that PCA shows the lowest L1-norm (7079.6) and L2- norm (7775). This indicates that the PCA will provide more sparsity for 2-D image compared to other sparsifying transforms.

5 Conclusion

In this research paper, we have presented an investigation on sparsity analysis of 1-D speech signal and 2-D image signal using different sparsity measurement metrics such as Gini Index, L1-norm and L2-norm. Furthermore, the Discrete Wavelet Transform (DWT) family, FFT, DCT, LPC and PCA are used as the sparsifying transforms.

Following major conclusions can be drawn based on the investigation:

- The result shows that the dmey wavelet (1-level decomposition) and the bior3.7 wavelet (3-level decomposition) show the greatest value of Gini index. The higher value of Gini index signifies the higher sparsity level of speech signal. Additionally, the bior5.5 wavelet (1-level and 3-level decomposition) and the bior3.1 (3-level decomposition) demonstrate the lowest value of the L1-norm and L2-norm. The lowest value of the L1-norm and L2-norm exhibits the higher sparsity of speech signal.
- Additionally, the DCT exhibits largest Gini index compared to FFT, LPC and PCA for speech signal. Also, the lowest value of the L1-norm and L2-norm are shown by the DCT.
- For 2-D image signal, the result shows that the bior3.7 (1-level decomposition) and the bior3.1 (3-level decomposition) exhibit the highest Gini index. Additionally, the bior5.5 wavelet (1-level and 3-level decomposition) and the bior3.1 (1-level decomposition) demonstrate the smallest value of the L1-norm and L2-norm.
- Moreover, the PCA exhibits the highest Gini index along with the lowest L1-norm and L2-norm which signifies the higher sparsity of 2-D image.
- Thus, the best sparsity basis can be selected based on sparsity measures such as Gini index, L1-norm and L2-norm.
- Furthermore, the Gini index can be considered as more consistent and general sparsity measure because of its advantages such as normalization within a range of 0 to 1, scale-invariant and independent of the total signal energy, compared to traditional norm metrics.

Signal sparsity is one of the basic and important features in compressed sensing (CS) and dictionary learning based applications. The sparse signal or compressible signal is the basic requirement for successful implementation of compressed sensing (CS) technique. Thus, sparsity may play an important role in different signal processing applications.

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