The numerical solution of an inverse two-phase stefan problem

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In this paper, the author studies an inverse two-phase Stefan problem obtained from a one-dimensional model of ice melting. This problem can be reduced to solve two hot-conduction equations. One is posed, so we can use the difference approximation method to solve it. The other is ill-posed. We first need to translate it into an integral equation, then use Tikhonov's method to regularize the integral equation.

Keywords: Inverse Two-Phase Stefan Problem; Difference approximation Method; Integral Equation Method; Volterra Integral Equations of the First kind; Tikhonov's Method.

1.Introduction

Problems with free boundaries concern a large number of physical phenomena of which many can be encountered in thermal industrial processes, such as casting, welding, purification by metal beams or laser machining by beams. In this paper, we consider an inverse two-phase Stefan problem which is from the following model.

The melting of a thin block of ice occupying an interval $b \le x \le L$ is described by the one-dimensional two-phase Stefan problem

$$\frac{\partial u_1}{\partial t} = {a_1}^2 \frac{\partial^2 u_1}{\partial x^2} \text{ in } 0 < x < s(t), t > 0 \tag{1}$$

$$\frac{\partial u_2}{\partial t} = {a_2}^2 \frac{\partial^2 u_2}{\partial x^2} \text{ in } s(t) < x < L, t > 0 \tag{2}$$

along with boundary conditions

$$u_1(s(t),t) = u_2(s(t),t) = 0 \tag{3}$$

$$s'(t) = -\lambda_1 \frac{\partial u_1}{\partial x}(s(t),t) + \lambda_2 \frac{\partial u_2}{\partial x}(s(t),t) \quad t > 0, \tag{4}$$

$$u_1(0,t) = v(t) \quad t > 0, \tag{5}$$

$$\frac{\partial u_2}{\partial x}(L,t) = 0 \quad t > 0, \tag{6}$$

and initial data

$$\begin{gathered} s(0) = b, \ \ 0 < b < L, \\ u_1(x,0) = u_1^0(x), \ \ 0 < x < b, \\ u_2(x,0) = u_2^0(x), \ \ b < x < L, \end{gathered} \tag{7}$$

in which u_1, u_2 represent the state functions of process and s the position of free boundary, $a_1, a_2, \lambda_1, \lambda_2$ are positive constants and $a_1 \neq a_2, \lambda_1 \neq \lambda_2$.

Given the initial distributions $u_1^0, \ \ u_2^0$ with compatible conditions

$$u_1^0(b) = u_2^0(b) = 0, \ \ \frac{\partial}{\partial x} u_2^0(L) = 0 \tag{8}$$

the so-called inverse Stefan problem is to find, for a prescribed interface s ,a time dependent function v such that the problem (1)-(8) has continuous solutions u_1, u_2 on $0 \leq x \leq s(t), t \geq 0$ and $s(t) \leq x \leq L, t \geq 0$ respectively.

We have a lot of numerical methods to solve the inverse one-phase Stefan problem .In paper[2],the problem is reduced to a system of integral equation. Jochum considers the inverse Stefan problem as a problem of nonlinear approximation theory(see[3,4]).In paper[5],the author use Adomian decomposition method to solve this problem .In paper[6],for solution of one phase two-dimensional problems, authors use a complete family of solutions to the heat equation to minimize the maximal defect in the initial-boundary data.

There are few numerical methods to solve the inverse two-phase Stefan problem. So the purpose of this paper is to construct a numerical method to the inverse two-phase Stefan problem. The problem (1) - (8) can be reduced to solve two heat-conduction equations.

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = a_1^{\,2} \dfrac{\partial^2 u_1}{\partial x^2}, & 0 < x < s(t), t > 0 \\ u_1(x,0) = u_1^0(x), & 0 \leq x \leq b = s(0) > 0 \\ u_1(0,t) = v(t), & t > 0 \\ u_1(s(t),t) = 0, & t > 0 \end{cases} \tag{9}$$

$$\begin{cases} \dfrac{\partial u_2}{\partial t} = {a_2}^2 \dfrac{\partial^2 u_2}{\partial x^2}, & s(t) < x < L, t > 0 \\ u_2(x,0) = u_2^0(x), & b < x < L, \\ \dfrac{\partial u_2}{\partial x}(L,t) = 0, & t > 0 \\ u_2(s(t),t) = 0, & t > 0 \end{cases} \tag{10}$$

$$s'(t) = -\lambda_1 \frac{\partial u_1}{\partial x}(s(t),t) + \lambda_2 \frac{\partial u_2}{\partial x}(s(t),t), \qquad t > 0. \tag{11}$$

2.Main Calculation Process

We first need to translate Eq. (11) into an integral equation, Then the inverse problem (9)-(11) can be translated to an integral equation problem.

Let $G(x,t;\xi,\tau) = E(x-\xi, a_1^2(t-\tau)) - E(x+\xi, a_1^2(t-\tau))$, $\tilde{G}(x,t;\xi,\tau) = E(x-\xi, a_1^2(t-\tau)) + E(x+\xi, a_1^2(t-\tau))$..

In $0 < \varepsilon < \tau < t-\tau, 0 < \xi < s(\tau)$ integrating Green identical equation, the following equation can be obtained. $a_1^2 \dfrac{\partial}{\partial \xi}(G \dfrac{\partial u_1}{\partial \xi} - u_1 \dfrac{\partial G}{\partial \xi}) - \dfrac{\partial}{\partial \tau}(G u_1) = 0,$ And then let $\varepsilon \to 0.$ Through a simple calculation, it is rewritten as

$$u_1(x,t) = \int_0^b G(x,t;\xi,0) u_1^0(\xi) d\xi + a_1^2 \int_0^t v(\tau) \frac{\partial G}{\partial \xi}(x,t;0,\tau) d\tau$$

$$+ a_1^2 \int_0^t \frac{\partial u_1}{\partial \xi}(s(\tau),\tau) G(x,t;s(\tau),\tau) d\tau \qquad 0 < x < s(t), 0 < t < T \tag{12}$$

Eq. (12) about x derivation on both ends, and let $x \to s(t) - 0$ .It can be rewritten as

$$a_1^2 \int_0^t v(\tau) \tilde{G}_\tau(s(t),t;0,\tau) d\tau = (1 - \frac{a_1^2}{2}) \omega_1(t) - \int_0^b u_1^0(\xi) G(s(t),t;\xi,0) d\xi$$

$$- a_1^2 \int_0^t \omega_1(t) \frac{\partial G}{\partial x}(s(t),t;s(\tau),\tau) d\tau \tag{13}$$

Let

$$\overline{K}(t,\tau)=\tilde{G}_{\tau}(s(t),t;0,\tau),$$

$$\overline{g}(t)=\frac{1}{a_1^2}[(1-\frac{a_1^2}{2})\omega_1(t)-\int_0^b u_1^0(\xi)G(s(t),t;\xi,0)d\xi$$

$$-a_1^2\int_0^t\omega_1(t)\frac{\partial G}{\partial x}(s(t),t;s(\tau),\tau)d\tau]$$

So Eq. (13) can be written as

$$\int_0^t\overline{K}(t,\tau)v(\tau)d\tau=\overline{g}(t) \tag{14}$$

In order to solve the inverse problem (9)-(11), we just need to work out $v(t)$ from the above integral equation.

Eq. (14) is a Volterra integral equation of first kind. Solving this kind equation is an ill-posed problem. So we need to use Tikhonov's method to regularize the integral equation.

Let $\tilde{K}(t,\tau)=\begin{cases}\overline{K}(t,\tau), & 0<\tau<t<T\\ 0, & 0<t<\tau<T\end{cases}$

Defining J is an integral operator form $L_2(0,T)$ to $L_2(0,T)$.

$Jv(t)=\int_0^t\tilde{K}(t,\tau)v(\tau)d\tau,$ where $v(t)\in L_2(0,T).$

Therefore Eq. (14) can be written as: $Jv(t)=\overline{g}(t).$

Assuming $\left|\overline{g}^{\delta}(t)-\overline{g}(t)\right|\le\delta,$ so Eq.(14)can be translate to the following equation:

$$\alpha v_{\alpha}^{\delta}+J'Jv_{\alpha}^{\delta}=J'\overline{g}^{\delta}, \tag{15}$$

Where α is a given constant. J' is an adjoint operator of J.

$J'v(t)=\int_0^T\tilde{K}(t,\tau)v(\tau)d\tau,$ where $v(t)\in L_2(0,T).$

In this calculation, the operator of regularization solution and the regularization solution are $R_{\alpha}=(\alpha I+J'J)^{-1}J',\ v_{\alpha}^{\delta}=R_{\alpha}\overline{g}^{\delta}=(\alpha I+J'J)^{-1}J'\overline{g}^{\delta}.$

Regularization equation is

$$\alpha v(t)+\int_t^T\int_0^{\xi}\overline{K}(\xi,t)\overline{K}(\xi,\tau)v(\tau)d\tau d\xi=\int_t^T\overline{K}(\tau,t)\overline{g}(\tau)d\tau \tag{16}$$

This equation is a posed Volterra integral equation of second kind. From this equation, the approximate solution of $v(t)$ can be solved as $v_{\alpha}^{\delta}(t)$. This problem is posed. In order to solve Eq. (16), we translate the double integral equation to a repeated integral equation. It can be rewritten as:

$$\alpha v(t)+\int_0^t K_1(t,\tau)v(\tau)d\tau+\int_t^T K_2(t,\tau)v(\tau)d\tau=g_1(t) \tag{17}$$

Then divide the interval $[0,T]$. $t_k=k\Delta t,\ k=0,1,2,\ldots,N.$ To apply the left rectangle formula to Eq. (17), then the following equation can be obtained.

$$\begin{cases}\alpha v_k+\sum\limits_{j=0}^{k-1}K_{k,j}^{(1)}v_i\Delta t+\sum\limits_{j=k+1}^{N\text{-}1}K_{k,j}^{(2)}v_i\Delta t=g_k^{(1)}, \quad k=0,1,2,\ldots,N-1,\\ v_N=0\end{cases}$$

Since $\overline{K}(t,\tau)=\widetilde{G}(s(t),t;0,\tau)$, so

$$\overline{K}(t_k,t_j)=\frac{1}{2a_1\sqrt{a_1\pi}(t_k-t_j)^{3/2}}\exp[-\frac{s^2(t_k)}{4a_1^2(t_k-t_j)}]$$

$$-\frac{1}{4a_1{}^2\sqrt{a_1\pi}(t_k-t_j)^{5/2}}\exp[-\frac{s^2(t_k)}{4a_1^2(t_k-t_j)}]$$

$$K_1(t_k,t_j)=\int_{t_k}^T\overline{K}(\xi,t_k)\overline{K}(\xi,t_j)d\xi\approx\sum_{i=k+1}^N\overline{K}(t_i,t_k)\overline{K}(t_i,t_j)\Delta t=K_{k,j}^{(1)}$$

$$K_2(t_k,t_j)=\int_{t_j}^T\overline{K}(\xi,t_k)\overline{K}(\xi,t_j)d\xi\approx\sum_{i=j+1}^N\overline{K}(t_i,t_k)\overline{K}(t_i,t_j)\Delta t=K_{k,j}^{(2)}$$

The right rectangle formula for $g_1(t_k)=\int_{t_k}^T\overline{K}(\tau,t_k)\overline{g}(\tau)d\tau,$ we can obtain

$$g_k^{(1)}=\sum_{i=k+1}^N\overline{K}(t_i,t_k)\overline{g}_i\Delta t,$$

By discretization

$$\overline{g}(t)=\frac{1}{a_1^2}[(1-\frac{a_1^2}{2})\omega_1(t)-\int_0^b u_1^0(\xi)G(s(t),t;\xi,0)d\xi$$

$$-a_1^2\int_0^t\omega_1(t)\frac{\partial G}{\partial x}(s(t),t;s(\tau),\tau)d\tau]$$

$\overline{g}_i$ is obtained as

$$\overline{g}_i = \frac{1}{a_1^2}[(1-\frac{a_1^2}{2})\overline{\omega}_k - \sum_{j=0}^{p} u_1^0(z_j)G(s(t_i,t_i,z_j,0))\Delta z$$
$$- a_1^2\sum_{j=0}^{i-1}\overline{\omega}_k G_x(s(t_i),t_i,s(t_j),t_j)\Delta z]$$

Where $p=\frac{b}{\Delta z}, z_j = j\Delta z$.

$$G(s(t_i),t_i,z_j,0) = \frac{1}{2a_1\sqrt{\pi t_i}}\exp[-\frac{(s(t_i)-z_j)^2}{4a_1^2 t_i}]$$
$$-\frac{1}{2a_1\sqrt{\pi t_i}}\exp[-\frac{(s(t_i)+z_j)^2}{4a_1^2 t_i}],$$

$$G_x(s(t_i),t_i,s(t_j),t_j) = \frac{1}{2a_1\sqrt{\pi(t_i-t_j)}}\exp[-\frac{(s(t_i)-s(t_j))^2}{4a_1^2(t_i-t_j)}](-\frac{s(t_i)-s(t_j)}{2a_1^2(t_i-t_j)})$$
$$-\frac{1}{2a_1\sqrt{\pi(t_i-t_j)}}\exp[-\frac{(s(t_i)+s(t_j))^2}{4a_1^2(t_i-t_j)}](-\frac{s(t_i)+s(t_j)}{2a_1^2(t_i-t_j)})$$

Here we seek the value of u_1 .Since the value of $v(t)$ is obtained so from Eq. (12) the value of u_1 can be obtained. In order to obtain the discrete value of u_1, we make a section of the rectangular area $A=\{(x,t)|0\le x\le L, 0\le t\le T\}$ .From Eq. (9), we can obtain the original value of u_1 is $u_1(x_j,0)=u_1^0(x_j), j=1,2,\ldots,n=\left[\frac{b}{\Delta x}\right]$. The boundary value of u_1 is $u_1(0,t_k)=v(t_k), u_1(s(t_k),t_k)=0, \quad k=0,1,2, \ldots, N.$

When $(x_j,t_k)\in C=\{(x,t)|0<x<s(t), 0<t<T\}$, the left rectangle formula is applied to Eq. (12).The following result can be obtained

$$\overline{u}_j^k = \Delta x\sum_{i=0}^{n}\tilde{u}_i G_{j,i}^{k,0} + a_1^2\Delta t\sum_{l=0}^{k} v_l \frac{\partial}{\partial\xi}G_{j,0}^{k,l} + a_1^2\Delta t\sum_{l=0}^{k}\overline{\omega}_l G_{j,s_l}^{k,l}.$$

Where $\tilde{u}_i = u_1^0(\xi_i)$, $G_{j,i}^{k,l} = G(x_j,t_k;\xi_i,\tau_l)$, $G_{j,s_l}^{k,l} = G(x_j,t_k;s(t_l),\tau_l)$, $\frac{\partial}{\partial\xi}G_{j,0}^{k,l} = G_\xi(x_j,t_k;0,\tau_l)$, $i=0,1, \ldots, n$, $l=0,1, \ldots, k$. So $\overline{u}_j^k$ is the difference approximation of $u_1(x_j,t_k)$.

References

1. D.D.Ang, A.Pham Ngoc and D.N.Thanh, Regularization of an inverse two-phase Stefan problem, Nonlinear Anal.34, 719-731 (1998).
2. D.D.Ang, A.Pham Ngoc and D.N.Thanh, Abidimensional inverse Stefan problem: identification of boundary value , J.Comput.Appl. Math.80, 227-240,(1997).
3. P.Jochum, The inverse Stefan problem as a problem of nonlinear approximation theory, J.Approx. Theory 30, 81-98, (1980).
4. P.Jochum, The numerical solution of the inverse Stefan prob-lem, Numer.Math.34, 411-429,(1980).
5. R.Grzymkowski and D.Slota, One-phase inverse Stefan problem solved by Adomian decomposition method, Comput.Math.Appl.51, 33-40, (2006).
6. D.Colton and R.Reemtsen, The numerical solution of the inverse Stefan problem in two space variables, SIAM J.Appl.Math.44,996-11013,(1984).
7. Zhijie GU and Yongji Tan, Fundamental solution method for inverse source problem of plate equation, Applied Mathematices and Mechanics, (2012).
8. Kyung-Beom Yoon ,Inverse Problem of Flames Surface Properties of Wood using a Repulsive Pariticle Swarm Optimization Algorithm, Journal of Thermal Science,(2015).