The Scale-free Behaviors of Random SA-mixed Network Models

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We constructed the random SA-mixed models based on Sierpinski and 2-dimension Apollonian networks, which having scale-free properties. In this paper, we prove the scale-free of random SA-mixed network model by two of methods including statistical parameter and partial differential equation, we get the same consequence of scale-free by different ways, that is degree distribution follows a power law.

Keywords: Scale-free; Sierpinski networks; Apollonian networks; Degree cumulative distribution.

1. Introduction

In studies of networks citations between scientific papers. Derek de Solla Price showed in 1965 that the number of links to papers i.e., the number of citations they receive had a heavy-tailed distribution following a Pareto distribution or power law, and thus that the citation network is scale-free. Moreover, not all vertices in a network have the same number of edges, the spread in the number of edges of the diverse vertices, or a vertices's degree, is characterized by the degree distribution $P(k)$ which gives the probability that a randomly selected vertice has exactly k edges, and it's degree distribution obey $P(k)\sim k^{-\gamma}$, where γ is a power law index falling into the range $1<\gamma<3$. In people daily life, there are too many real networks having scale-free feature, for example WWW (Ref. [1]), the airplanes connection networks, metabolic and protein networks (Ref. [2], [3]) and so on.

The dynamical properties of the scale-free model can be addressed using various analytic approaches. The cintinuum theory proposed in (Ref. [4], [5]) focuses on the dynamics of vertice degrees. Widely used are the master equation approach of Dorogovtsev, Mendes and Samukhin (Ref. [6]) and the rate equation appriach introduced by Krapivsky, Redner and Leyvraz (Ref. [7]). We will focus on the continuum theory. Barabási and Albert proposed a generative mechanism to explain the appearance of power-law distributions, which they called “growth” and “preferential attachment”. Because of the dynamical evolution of most real-life networks is random, so we establish a dynamical equation to explore networks. We validate random SA-mixed network models scale-free by computing partial differential equation to maximum model and minimum model of random SA-mixed network model, model, the number of vertices and edges is maximal or minimal, called maximum model and minimum model respectively. This is a kind of new way to research dynamic networks.

In this paper, we adopt two methods to prove the scale-free of our random SA-mixed network models, and apply the degree preferential attachment mechanism (Ref. [7]) to make the models be scale-free and connect one part new adding vertices with edges to form masses of higher dense triangles for letting it be small-world.

2. SA-mixed models

We define some fractal operations for building our models, and some fractal operations can be found in [8]. Let $S(0)$ be a graph having three vertices A,B,C and three edges such that any pair of vertices is joined by an edge, we call $S(0)$ a triangle (see Figure 1.(a) and (b)). Clearly, the triangle $S(0)$ divides the plane into two parts: One is out of $S(0)$, called the **outer face** of $S(0)$; one is inside $S(0)$, called the **inner face** of $S(0)$.

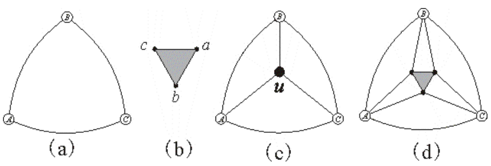

Figure 1: (a) S(0); (b) an operation triangle; (c) (I)-operation; (d) (II)-operation.

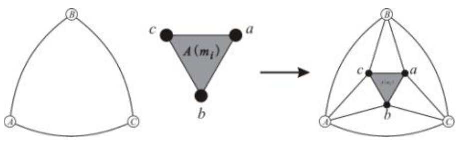

Figure 2: A fractal (III)-operation.

A **fractal (I)-operation** yields a configuration $H_{(1)}$ (see Figure 1.(c)) in the way: Add a vertex u into the inner face of a triangle shown in Figure 1.(a), and then join u with three vertices A, B and C by three edges, respectively. We define a labelling function g for $H_{(1)}$ as: $g(A)=g(B)=g(C)=k-1$, $g(u)=k$ for $k\geq 1$. A f**ractal (II)-operation** produces a complex configuration $H_{(2)}$ (see Figure 1.(d)) in the way: Put an *operation triangle* abc shown in Figure 1.(b) into the inner face of an *objective triangle ABC* shown in Figure 1.(a), and furthermore join c with A and B by two edges, join a with B and C by two edges, and then join b with C and A by two edges. Clearly, $H_{(2)}$ has 7 inner faces and an outer face. We define a labelling function f of $H_{(2)}$ as: $f(A)=f(B)=f(C)=k-1$, $f(a)=f(b)=f(c)=k$ for

$k \geq 1$. A **fractal (III)-operation** is a mixed operation of the fractal (I)-operation and the fractal (II)-operation. We label the three vertices of the outer face of an Apollonian model $A(m_i)$ with *a*, *b*, *c*, and put this $A(m_i)$ into the inner face of an objective triangle *ABC*, and do a fractal (II)-operation (see Figure 2). Since the Sierpinski model $S(j)$ has 6^{j-1} objective triangle *ABC* with $f(A)=f(B)=f(C)=j$ at time step *j*, we can set the initial SA-mixed model $S(t_0, A(m_i|_1^n))$ by $6^{t_0-1} \geq n$.

2.1. *Sierpinski and Apollonian models*

The Sierpinski model $S(t)$ and the Apollonian model A(t) can be generated by the Sierpinski-algorithm and Apollonian-algorithm respectively.

Sierpinski-algorithm. ***Initialization.*** At time step t=0, $S(0)$ is shown in Figure 3, and the labelling function *f* holds $f(A)=f(B)=f(C)=0$. ***Iteration***. At time step *t*, do a fractal (II)-operation to every objective triangle *abc* of $S(t-1)$ with no $f(a)=f(b)=f(c)$ and at least one of three labels $f(x)$, $f(y)$ and $f(z)$ is equal to $t-1$, and label three vertices of each operation triangle with *t* under *f*.

Apollonian-algorithm. ***Initialization.*** At time step t=0, $A(0)$ is a triangle *ABC*, and the labelling function *f* holds $f(A)=f(B)=f(C)=0$. ***Iteration.*** At time step *t*, do a fractal (I)-operation to every triangle *xyz* of $A(t-1)$ with at least the number of three labels $f(x)$, $f(y)$ and $f(z)$ is equal to $t-1$, and label the vertex added with *t* under *f*.

According to [8] and [9], $S(t)$ is equal to $P_{cum}(k)=6(k-1)^{1-\gamma S}$ with a power law exponent $\gamma_S = 2 + \ln 2/\ln 3$. Thus, $S(t)$ is scale-free. The cumulative distribution of $A(t)$ is $P_{cum}(k) \propto 3^{1-\ln 3/\ln 2} k^{1-\gamma_A}/2$ with $\gamma_A = 1 + \ln 3/\ln 2$, they are scale-free networks.

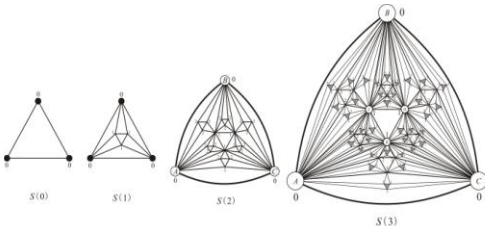

Fig. 3 The construction of the Sierpinski model $S(t)$ at the first four steps.

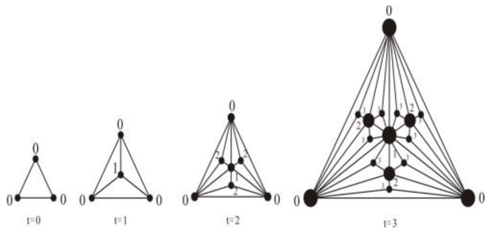

Fig 4 The construction of the Apollonian networks at the first four steps.

2.2. *SA-mixed model* $S(t, A(m_i|_1^n))$

An SA-mixed model $S(t, A(m_1), A(m_2), \ldots, A(m_n))$ can be built up by the SA-mixed-algorithm **SA-algorithm**. For *n* Apollonian models $A(m_1)$, $A(m_2)$, …, $A(m_n)$, we arrange the vertex numbers of $A(m_i)$ with i=1,2, …, *n* in an ordered order from small to large, in other words, the vertex number of $A(m_j)$ does not exceed that of $A(m_{j+1})$ for j=1,2, …, $n-1$, we call them ***motifs***.

SA-mixed-algorithm. ***Initialization***. At time step t_0, $S(t,A(m_i|_1^n))$ is obtained by choose randomly $A(m_1)$, $A(m_2)$, …, $A(m_n)$ and put randomly them into 6^{t_0-1}. objective triangles in which every triangle *ABC* holds $f(A)=f(B)=f(C)=t_0$. ***Iteration***. At time step *t*, do a fractal (III)-operation to every *objective triangle ABC* of $S(t)$. The resulting model is just an SA-mixed model $S(t, A(m_1), A(m_2), \ldots, A(m_n))$.

We write $S(t,A(m_i|_1^n))$ instead of $S(t, A(m_1), A(m_2), \ldots, A(m_n))$ for short. [17] have computed the degree spectrum of $S(t,A(m_i|_1^n))$. For the sake of simplicity, if all motifs $A(m_i)$ added in implementing fractal (III)-operations to all objective triangles of $S(t)$ are equal to $A(m_1)$ at time step *t*, we get a particular SA-mixed model $S_{\min}(t,A(m_i|_1^n))$. Similarly, another particular SA-mixed model $S_{\max}(t,A(m_i|_1^n))$ can be obtained when all motifs $A(m_i)$ added in implementing fractal (III)-operations to all objective triangles of $S(t)$ are equal to $A(m_n)$ at time step *t*. In order to simplify the calculation, we let $n_v(t)$ and $n_e(t)$ be the total numbers of vertices and edges of $S(t,A(m_i|_1^n))$ at step time *t*, respectively; and let $n_v^{\min}(t)$ and $n_e^{\min}(t)$ be the total numbers of vertices and edges of $S_{\min}(t,A(m_i|_1^n))$ at step time *t*, respectively; and let $n_v^{\max}(t)$ and $n_e^{\max}(t)$ be the total numbers of vertices and edges of $S_{\max}(t,A(m_i|_1^n))$ at step time *t* respectively. Obviously, we have

$$n_v^{\min}(t)=[6^{t-1}(5\cdot 3^{m_1}+61)+24]/10,\ n_e^{\min}(t)=[^{t-1}(5\cdot 3^{m+1_1}+123)+12]/10.$$

The average vertex degree $\langle k\rangle_{\min}$ of $S_{\min}(t,A(m_i|_1^n))$ is equal to $2n_e^{\min}(t)/n_v^{\min}(t)\to 6$ as t approaches to infinity. Therefor, $S_{\min}(t,A(m_i|_1^n))$ is sparse. By

$$n_v^{\max}(t)=[6^{t-1}(5\cdot 3^{m_n}+61)+24]/10,\ n_e^{\max}(t)=[6^{t-1}(5\cdot 3^{m_n+1}+123)+1]/10,$$

the average vertex degree $\langle k\rangle_{\max}$ of $S_{\max}(t,A(m_i|_1^n))$ is equal to $2n_e^{\max}(t)/n_v^{\max}(t)\to 6$ as $t\to\infty$, which means that $S_{\max}(t,A(m_i|_1^n))$ is also sparse. We guess: *Each SA-mixed model* $S(t,A(m_i|_1^n))$ *is sparse*.

2.3. *Statistical parameter*

To show our models having scale-free nature, in the following we concentrate on the degree cumulative distribution $P_{cum}(k)$ defined in [14] to measure the degree distribution $P(k)$ of the SA-mixed model $S(t,A(m_i|_1^n))$, where the degree distribution $P(k)$ is the probability that a randomly selected vertex has exactly *k* edges in $S_{\max}(t, A(m_i|_1^n))$. For estimating the topological structure of $S(t, A(m_i|_1^n))$, we compute the degree cumulative distributions $P_{cum}^{\min}(k)$, $P_{cum}^{\max}(k)$ and the degree distributions $P_{min}(k)$, $P_{max}(k)$ of two SA-mixed models $S_{\min}(t, A(m_i|_1^n))$ and $S_{\max}(t, A(m_i|_1^n))$, respectively, in the following argument. Since the degree spectrums of two models $S_{\min}(t, A(m_i|_1^n))$ and $S_{\max}(t, A(m_i|_1^n))$ *are* discrete, so we can use a technique introduced by Newman in (Ref. [13], [14]). Dorogovstev defined the cumulative distribution $P_{cum}(k) = [n_v(t)]^{-1} \sum_{k' \geq k} |V(k',t)| \propto k^{1-\lambda}$, where $|V(k',t)|$ is the number of vertices of degree k' (Ref. [14]). Here, we get the cumulative distribution of $S_{\min}(t, A(m_i|_1^n))$ as follows

$$P_{cum}^{\min}(k) = \frac{1}{n_v(t)} \sum_{k' \geq k} |V(k',t)| = \frac{36 \cdot 6^{\tau-1} + 5 \cdot 3^{m_1}}{6^{t-1}(5 \cdot 3^{m_1+1} + 123)}$$

where τ is the time that vertices with degree k added in the SA-mixed model, plugging $\tau = t + 2 - \ln(k - 2^{m_1+1} + 2^{m_1}) / \ln 3$ into the above equation, and when t is large enough,

$$P_{cum}^{\min}(k) \approx \frac{1290}{5 \cdot 3^{m_1+1} + 123} \cdot (k - 2^{m_1+1} + 2^{m_1})^{-1-\frac{\ln 2}{\ln 3}} \approx \frac{259}{3^{m_1} + 1} k^{-1-\frac{\ln 2}{\ln 3}}$$

According to [14], it is convenient to obtain that the degree distribution via its cumulative degree distribution.

$$P_{cum}^{\min}(k) = -\frac{\partial P_{cum}^{\min}(k)}{\partial k} \approx \frac{259}{3^{m_1} + 1} \left(1 + \frac{\ln 2}{\ln 3}\right) k^{-2-\frac{\ln 2}{\ln 3}}$$

Thereby, the degree distribution $P_{min}(k)$ follows a power law form with the exponent γ=2+ln2/ln3, where 2<γ<3. It indicates that the minimal model of random SA-mixed models is scale-free. For the model $S_{\max}(t, A(m_i|_1^n))$, we have

$$P_{cum}^{\max}(k) = \frac{1}{n_v(t)} \sum_{k' \geq k} |V(k',t)| \sum_{k' \geq k} = \frac{36 \cdot 6^{\tau-1} + 5 \cdot 3^{m_n}}{6^{t-1}(5 \cdot 3^{m_n+1} + 123)}$$

plugging $\tau = t + 2 - \ln(k - 2^{m_n+1} + 2^{m_n}) / \ln 3$ into the above equation, and when t is large enough, we obtain

$$P_{cum}^{\max}(k) \approx \frac{1290}{5 \cdot 3^{m_n+1} + 123} \cdot (k - 2^{m_n+1} + 2^{m_n})^{-(1+\frac{\ln 2}{\ln 3})} \approx \frac{259}{3^{m_n} + 1} k^{-(1+\frac{\ln 2}{\ln 3})}.$$

Similarly with $P^{\max}(k) = -\partial P_{cum}^{\max}(k) / \partial k$, we can easily compute the degree distribution $P^{\max}(k) \propto k^{-\gamma}$, 2<$\gamma$=2+ln2/ln3<3, where C is a constant. So, the maximal model of random SA-mixed models is also scale-free. Due to the degree distribution of the maximum model and the minimum model are all obey the power law distribution, we can use this two models to approximate the random SA-mixed models. Obviously, we have $P_{\min}(k) \leq P(k) \leq P_{\max}(k)$, which means that each random SA-mixed model also follows a certain power law distribution, so it is scale-free.

2.4. *Continuum theory*

Barabási et al. [5] introduce the continuum approach that calculates the time dependence of the degree k_i of a given vertice i, this degree will increase every time a new vertice enters the system and links to node i, the probability of this process being $\prod(k_i)$. Assuming that k_i is a continuous real variable, the rate at which k_i changes is proportional to $\prod(k_i)$, and k_i satisfies the dynamical equation. Ma *et al.* (ref. [16]) show a dynamic differential equation $\frac{\partial k_i(t)}{\partial t} = f^*(t) + g^*(t) + h^*(t) + z^*(t) + \varphi(t)$. For minimal random SA-mixed model, others functions $g^*(t) = h^*(t) = z^*(t) = \varphi(t) = 0$, k_i satisfies the dynamical equation

$$\frac{\partial k_i(t)}{\partial t} = 6^{t-1} \cdot \frac{3^{m_1}+5}{2} \cdot 6^{t-1} \cdot \frac{3(3^{m_1}+1)}{2} \prod(k_i) = B \cdot 6^{2t} \cdot \frac{k_i}{\sum k_j}$$

where $B = (3^{m_1}+1)(3^{m_1}+5)/48$, the sum in the denominator goes over all vertices in the system except the newly introduced one, thus its value is $\sum_j k_j = 2n_e(t) = \sum_{i=0}^{t} B \cdot 6^{2i} = (6^{2t+2}-1)/35$. The solution of this equation, with the initial condition $k_i(t_i) = B \cdot 6^{2t_i}$, is $k_i(t) = B \cdot 6^{2t_i} \cdot e^{t-t_i}$, it indicates that the degree of all vertices evolves the same way, following a power-law. Using above equation, the probability that a vertice has a degree $k_i(t)$ smaller than k, $P(k_i(t)<k)$ can be written as

$$P(k_i(t) < k) = P(t_i > -(\ln k - \ln B)/3 + t/3) = 1 - P(t_i \le -(\ln k - \ln B)/3 + t/3)$$

Assuming that we add the vertices at equal time intervals to the network, the t_i values have a constant probable density $P(t_i) = (B \cdot 6^{2t_i} + t_i)^{-1}$. The degree distribution $P(k)$ can be computed as

$$P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = \frac{t}{9k(B \cdot 6^{2t} + t)[-\frac{1}{3}(\ln k - \ln B) + \frac{1}{3}t]^2}$$

predicting that $P(k) \propto k^{-\gamma}$ with $1<\gamma<2$. For maximal random SA-mixed model, we also get the same consequence and we have proven the scale-free of random SA-mixed model.

3. Conclusion

For the scale-free of network models, based on the random SA-mixed network model, we use two methods to explored and proved the scale-free characteristics. Different ways to get the same conclusion, one is statistical parameter, another is continuum theory, we set up the general partial differential equation with five characteristics functions and achieve its solutions by abstract numerical analysis under a special initial condition. In the dynamical evolution process of real-life network, it's necessary for people for research it's mechanism, so we must to look for more methods to research networks.

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