A Hilbert Type Inequality for Finite Series

Baoju Sun
Department of Mathematics
Zhejiang University of Water Resources and Electric Power, Hangzhou, Zhejiang 310018, China
sunbj@zjweu.edu.cn

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Abstract. In this paper, by using Cauchy inequality, monotonicity of functions, Hilbert type inequality with finite series version is established.

Introduction

If $a_n \ge 0, b_n \ge 0$, $m, n \in \mathrm{N}$, such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty, 0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then the well known Hilbert inequalities are given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < p\left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}. \quad (1)$$

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{a_m b_n}{m+n} < p\left(\sum_{n=1}^{N} a_n^2\right)^{1/2}\left(\sum_{n=1}^{N} b_n^2\right)^{1/2}. \quad (2)$$

Hilbert type inequalities are given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{max(m,n)} < 4\left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}. \quad (3)$$

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{a_m b_n}{max(m,n)} < 4\left(\sum_{n=1}^{N} a_n^2\right)^{1/2}\left(\sum_{n=1}^{N} b_n^2\right)^{1/2}. \quad (4)$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left|\ln(m/n)\right| a_m b_n}{max(m,n)} < 8\left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}. \quad (5)$$

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{\left|\ln(m/n)\right| a_m b_n}{max(m,n)} < 8\left(\sum_{n=1}^{N} a_n^2\right)^{1/2}\left(\sum_{n=1}^{N} b_n^2\right)^{1/2}. \quad (6)$$

(see Hardy et al.[1]). In recently years, various improvements and extensions of the Hilbert inequality and Hilbert type inequalities appear in a great deal of papers (see [2-5]). Zhang xiaoming, Chu yuming ([2]) gave improvement of (2) as:

$$p^2\sum_{n=1}^{N} a_n^2 \sum_{n=1}^{N} b_n^2 - \left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{a_m b_n}{m+n}\right)^2 \ge \min_{1\le n\le N}\left\{n a_n^2\right\}\min_{1\le n\le N}\left\{n b_n^2\right\}\left[p^2\left(\sum_{n=1}^{N}\frac{1}{n}\right)^2 - \left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{1}{\sqrt{mn}(m+n)}\right)^2\right] \quad (7)$$

The major objective of this paper is to formulate new inequalities, which is improvement of (6).

Some lemmas

In order to prove our main result we need some lemmas, which we present in this section.

Lemma 1 If $1 \le n \le N, n, N$ are positive integers, then

(i)

$$\sum_{m=1}^{N}\frac{|\ln(m/n)|}{\sqrt{m}\cdot max(m,n)}<8n^{-1/2}. \quad (8)$$

(ii)

$$\sum_{m=1}^{N+1}\frac{|\ln(m/(N+1))|}{\sqrt{m}\cdot max(m,N+1)}<4(N+1)^{-1/2}. \quad (9)$$

(iii)

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{|\ln(m/n)|}{\sqrt{mn}\cdot max(m,n)}<8\sum_{n=1}^{N}\frac{1}{n}. \quad (10)$$

Proof. (i) see[7].

(ii)

$$\sum_{m=1}^{N+1}\frac{|\ln(m/(N+1))|}{\sqrt{m}\cdot max(m,N+1)}<\int_0^{N+1}\frac{|\ln(x/(N+1))|}{(N+1)\sqrt{x}}dx=\int_0^1\frac{-\ln t}{\sqrt{N+1}\sqrt{t}}dt=4(N+1)^{-1/2}.$$

(iii)

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{|\ln(m/n)|}{\sqrt{mn}\cdot max(m,n)}<\sum_{n=1}^{N}\frac{1}{\sqrt{n}}8n^{-\frac{1}{2}}=8\sum_{n=1}^{N}\frac{1}{n}.$$

Lemma 2 Let $f(N)=64\left(\sum_{n=1}^{N}\frac{1}{n}\right)^2-\left(\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{|\ln(m/n)|}{\sqrt{mn}\cdot max(m,n)}\right)^2$,

then $f(N+1)>f(N)$.

Proof. $f(N+1)-f(N)$

$$=64\left(2\sum_{n=1}^{N}\frac{1}{n}+\frac{1}{N+1}\right)-\left(\frac{2}{\sqrt{N+1}}\sum_{n=1}^{N+1}\frac{|\ln(n/(N+1))|}{\sqrt{n}\cdot max(N+1,n)}\right)\times\left[2\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|}{\sqrt{mn}\cdot max(m,n)}\right.$$

$$\left.+\frac{2}{\sqrt{N+1}}\sum_{n=1}^{N+1}\frac{|\ln(n/(N+1))|}{\sqrt{n}\cdot max(N+1,n)}\right]$$

$$>64\left(2\sum_{n=1}^{N}\frac{1}{n}+\frac{1}{N+1}\right)-\frac{8}{N+1}\left(16\sum_{n=1}^{N}\frac{1}{n}+\frac{8}{N+1}\right)=0.$$

Main results

Theorem 1 If $a_n\ge 0, b_n\ge 0, n=1,2,\ldots,N$. N is positive integer, then

$$64\sum_{n=1}^{N}a_n^2\sum_{n=1}^{N}b_n^2-\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|a_mb_n}{max(m,n)}\right)^2$$

$$\ge\min_{1\le n\le N}\{na_n^2\}\min_{1\le n\le N}\{nb_n^2\}\left[64\left(\sum_{n=1}^{N}\frac{1}{n}\right)^2-\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|}{\sqrt{mn}\cdot max(m,n)}\right)^2\right] \quad (11)$$

Proof. Let $c_n=\sqrt{n}a_n, d_n=\sqrt{n}b_n$, then inequality (11) is translated into

$$64\sum_{n=1}^{N}\frac{c_n^2}{n}\sum_{n=1}^{N}\frac{d_n^2}{n}-\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|c_m d_n}{\sqrt{mn}\cdot max(m,n)}\right)^2$$
$$\geq \min_{1\leq n\leq N}\{c_n^2\}\min_{1\leq n\leq N}\{d_n^2\}\left[64\left(\sum_{n=1}^{N}\frac{1}{n}\right)^2-\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|}{\sqrt{mn}\cdot max(m,n)}\right)^2\right]. \quad (12)$$

By using Cauchy inequality, we have

$$64\sum_{n=1}^{N}\frac{c_n^2}{n}\sum_{n=1}^{N}\frac{d_n^2}{n}-\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|c_m d_n}{\sqrt{mn}\cdot max(m,n)}\right)^2$$
$$\geq 64\sum_{n=1}^{N}\frac{c_n^2}{n}\sum_{n=1}^{N}\frac{d_n^2}{n}-\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|c_m^2}{\sqrt{mn}\cdot max(m,n)}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|d_n^2}{\sqrt{mn}\cdot max(m,n)}. \quad (13)$$

Let $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$

$$=64\sum_{n=1}^{N}\frac{c_n^2}{n}\sum_{n=1}^{N}\frac{d_n^2}{n}-\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|c_m^2}{\sqrt{mn}\cdot max(m,n)}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|d_n^2}{\sqrt{mn}\cdot max(m,n)}.$$

to compute the partial derivatives of a function $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$ with respect to c_i . d_i, and using lemma 1, we have

$$\frac{\partial f}{\partial c_l}=64\cdot\frac{2c_i}{i}\sum_{n=1}^{N}\frac{d_n^2}{n}-\sum_{n=1}^{N}\frac{2c_i|\ln(i/n)|}{\sqrt{i}\sqrt{n}\cdot max(i,n)}\sum_{n=1}^{N}\frac{d_n^2}{\sqrt{n}}\left(\sum_{m=1}^{N}\frac{|\ln(m/n)|}{\sqrt{m}\cdot max(m,n)}\right)$$
$$>\frac{128c_i}{i}\sum_{n=1}^{N}\frac{d_n^2}{n}-\frac{2c_i}{\sqrt{i}}\cdot 8i^{-1/2}\sum_{n=1}^{N}\frac{d_n^2}{\sqrt{n}}8n^{-1/2}=0..$$

thus $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$ is monotone increasing for c_i. In a similar way we can provde that $\frac{\partial f}{\partial d_i}>0,$ and this implies $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$ is monotone increasing for d_i. We obtain

$$f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$$
$$\geq f(\min_{1\leq n\leq N}\{c_n\},\min_{1\leq n\leq N}\{c_n\},\mathbf{L},\min_{1\leq n\leq N}\{c_n\},\min_{1\leq n\leq N}\{d_n\},\min_{1\leq n\leq N}\{d_n\},\mathbf{L},\min_{1\leq n\leq N}\{d_n\}).$$

In view of (13), (12) holds. The theorem is proved.

From theorem 1 and lammas 2, we have

Theorem 2 If $a_n\geq 0, b_n\geq 0, n=1,2,\mathbf{L},N.$ N is positive integer, then

$$64\sum_{n=1}^{N}a_n^2\sum_{n=1}^{N}b_n^2-\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{|\ln(m/n)|a_m b_n}{max(m,n)}\right)^2\geq 64\min_{1\leq n\leq N}\{na_n^2\}\min_{1\leq n\leq N}\{nb_n^2\}. \quad (14)$$

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