Half Quadratic Regularization Model for Image Restoration

Li Xuchao[1,a,*] , Ma Songyan [1,b] and Li Yuye[2]

[1]College of Computer and Information Engineering, Chifeng University, Inner Mongolia, China
[2]Department of Mathematics and Statistics, Chifeng University, Inner Mongolia, China
[a] bsx7096@sohu.com, [b] l6495312@sohu.com
*corresponding author

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Abstract. Image restoration easily smears image edge in unsteady area and produces stair effect in steady area. In order to overcome these shortcomings, the paper proposes a half-quadratic energy functional regularization model (EFRM) and structured Newton iterative algorithm. Firstly, for blurred image by system and Gaussian noise, fitting term is described by L_2 norm, regularization term is described by half-quadratic function, which can accurately describe image singular property, and the fitting term and regularization term constitute image restoration EFRM. Secondly, resort to Fenchel transform, by introducing auxiliary variables, the primal EFRM is converted into augmented image restoration EFRM. Finally, take advantage of preconditioned theory, Hessian matrix of the transformed model is structured, a new Newton project iterative algorithm is proposed. Comparing to several state-of-art approaches, numerical experiment results show that the proposed EFRM can effectively protect image edge and reduce stair effect, and show better visual effect and higher peak signal-to-noise ratio (PSNR).

1. Introduction

In science and industry, inverse problems arise in a variety of practical applications, such as compress sensing, computerized tomography imaging and biomedical imaging. These applications [1] for estimating some kinds of attributes of object, for example, in guiding system, we want to get high quality image of moving object. Unfortunately, because noise and parameters of imaging system are difficult to estimate, the measurement image is seriously blurred.

In order to obtain ideal image, construction of energy functional regularization model (EFRM) of image restoration, which is composed of fitting term and regularization term, is the most effective method [2]. Supposed that the measurement image is blurred by imaging system and Gaussian noise, according to the noise statistical distribution, the fitting term is described by L_2 norm [3], in order to embody singular property of image, the regularization term is characterized by total variation (TV) function space [1]. However, in steady area of image, TV can easily cause stair effect and produce false edges. For overcoming these shortcomings of TV regularization term, a lot of improved models are proposed, such as general TV model, adaptive TV model and fractional order

TV model [4], but the parameters of EFRM are difficult to determine. Although lifting order number of regularization term can reduce the stair effects, it seriously smears image edges. Thus, in image restoration processing, how to protect image edge in unsteady area and reduce stair effect in steady area is a dilemma problem. Mixture EFRM can solve the problem. However, the model is fourth order, which easily produces Gibbs phenomena [5].

In algorithm design, the semi-norm of TV is non-differential, traditional algorithm based on gradient is unable to use directly. TV semi-norm is approximated by smooth function, reference [1] proposed the steepest descent algorithm, but the convergence speed of the algorithm is slow. In order to accelerate the convergence speed of algorithm, using Fenchel transform, the primal EFRM is converted into differential dual model, reference [6] proposed gradient projection iterative algorithm. Using gradient of fitting term, reference [7] proposed fast iterative soft threshod algorithm (FISTA). However, the convergence speed of first order algorithm based on gradient is still slow. For overcoming the shortcoming, using second order differential smooth function to approximate TV semi-norm, design quasi-Newton projection iterative algorithm, BFGS algorithm and preconditioned conjugate gradient least square algorithm, however, Hessian matrix of EFRM is determind by fitting term and regularization term, the scale of Hessian matrix, without special structure, is large, and it is difficult to compute inverse matrix of Hessian matrix. In order to easily compute inverse Hessian matrix, singular value decomposition (SVD), general SVD (GSVD), diagonal and block diagonal Hessian matrix algorithms are proposed [3]. However, it is a difficult task for us to obtain diagonal Hessian matrix.

The rest of this paper is organized as follows : In section 2, we establish EFRM for restoration image blurred by system and Gausian noise. taking advantage of Fenchel transform, we convert primal EFRM into augmented EFRM, which is advantage of algorithm design. In section 3, using preconditioning theory, Hessian matrix of augmented EFRM is diagonalized, then a project Newton iterative algorithm is proposed. In section 4, in order to show the efficiency of the proposed model and algorithm, numerical experiments comparing the proposed with some recent algorithms are carried out. In section 5, conclusions and future works are given.

2. Half Quadratic Regularization Model

Supposed that ideal image and measurement image obey independent Gaussian distribution, the fitting term is described by L_2 norm, in order to protected image edge during the processing of image restoration, regularization term is described by half quadratic function, the EFRM of image restoration is formulated as

$$\boldsymbol{u}^* = \inf_{\boldsymbol{u}} \{ E(\boldsymbol{u}) \} \tag{1}$$

Where, $E(\boldsymbol{u}) = \|\boldsymbol{Au} - \boldsymbol{g}\|_{L_2(\Omega)}^2 + \lambda \int_\Omega \varphi(\boldsymbol{Du}) d\boldsymbol{x}$, $\inf\{\bullet\}$ represents lower limit, $\varphi(\bullet)$ is continuous and differential convex function, which satisfies linear growth condition, $\boldsymbol{D}$ is first order differential operator that represents image singular property. If the solution of (1) exists, then matrix $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ has inverse matrix, and operator $\boldsymbol{A}$ and $\boldsymbol{D}$ must satisfy the following relationship

$$\ker(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}) \cap \ker(\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D}) = \{0\} \tag{2}$$

Where $\mathrm{Ker}(\bullet)$ represents matrix kernel space. Because of the complex property of (1), it is unable to solve directly, using Fenchel transform [1], the potential function $\varphi(\bullet)$ is converted into dual function $\phi(\bullet)$, which is expressed as

$$\phi(\boldsymbol{t}) = \sup_{\boldsymbol{s}} \{ -\boldsymbol{ts}^2 + \varphi(\boldsymbol{s}) \} \tag{3}$$

Using convex dual relationship, $\varphi(\boldsymbol{s})$ is obtained from $\phi(\boldsymbol{t})$, $\boldsymbol{s}$ is substituted by $\boldsymbol{Du}$, carrying out simple deduction, plug (3) into (1), we get the augmented EFRM, which is expressed as

$$(\boldsymbol{u}^*,\ \boldsymbol{t}^*) = \inf_{\boldsymbol{u}} \{E(\boldsymbol{u},\ \boldsymbol{t})\} \tag{4}$$

Where $E(\boldsymbol{u},\ \boldsymbol{t}) = \|\boldsymbol{Au}-\boldsymbol{g}\|^2_{L_2(\Omega)} + \lambda\int_\Omega \sup_{\boldsymbol{t}}\left\{\left(\boldsymbol{t}(\boldsymbol{Du})^2\right)+\phi(\boldsymbol{t})\right\}dx$, dual variable $\boldsymbol{t}$ represents singular property of image. Formula (4) converts the solution of (1) into auxiliary variable and ideal image, and auxiliary variable represents image singular characteristics, the significant meaning of the transformation integrates signal filter processing into singular detection processing. The gradient of (4) about $\boldsymbol{u}$ and $\boldsymbol{t}$ is expressed as

$$\boldsymbol{GE}(\boldsymbol{u},\ \boldsymbol{t}) = \left[2\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{Au}-\boldsymbol{g})+2\lambda\boldsymbol{D}^{\mathrm{T}}\boldsymbol{tDu},\ \lambda\left((\boldsymbol{Du})^2+\phi'(\boldsymbol{t})\right)\right] \tag{5}$$

The Hessian matrix of (4) about $\boldsymbol{u}$ and $\boldsymbol{t}$ is expressed as

$$\boldsymbol{HE}(\boldsymbol{u},\ \boldsymbol{t}) = \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{bmatrix} = \begin{bmatrix} 2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}+2\lambda\boldsymbol{D}^{\mathrm{T}}\boldsymbol{tD} & 2\lambda\boldsymbol{D}^{\mathrm{T}}\boldsymbol{Du} \\ 2\lambda\boldsymbol{DuD} & \lambda\phi''(\boldsymbol{t}) \end{bmatrix} \tag{6}$$

Because of convex, lower semi-continuous function of $\varphi(\bullet)$, using Fenchel transform, the dual potential function is also convex, lower semi-continuous function, hence $\phi''(\boldsymbol{t}) \geq 0$. According to (2), formula (2) about $\boldsymbol{u}$ and $\boldsymbol{t}$ is positive, hence, the solution of (4) is unique.

3. Computational Theory of Augmented EFRM

3.1. Classical Newton Iterative Algorithm

If the scale of Hessian matrix is not large, and the inverse matrix is easily computed, we can obtain the solution of augmented EFRM by classical Newton algorithm, which is expressed as following

1) Set maximum iterative number N, $\boldsymbol{d}_0 = [\boldsymbol{u}_0, \boldsymbol{t}_0]^{\mathrm{T}}$.
2) Compute (5) for gradient of augmented EFRM, and compute (6) for Hessian matrix.
3) Compute search direction of Newton iterative algorithm, which is expressed as

$$\boldsymbol{\theta} = [\boldsymbol{HE}(\boldsymbol{u},\boldsymbol{t})]^{-1}\boldsymbol{GE}(\boldsymbol{u},\boldsymbol{t}) \tag{7}$$

4) Using linear search algorithm for getting search step length, which is defined as

$$\tau_k = \inf_{\tau>0}\{E(\boldsymbol{d}_k+\tau\boldsymbol{\theta})\} \tag{8}$$

5) Update the solution of augmented EFRM, which is written as

$$\boldsymbol{d}_{k+1} = \boldsymbol{d}_k + \tau_k\boldsymbol{\theta} \tag{9}$$

6) If $\left\|E(\boldsymbol{u}^{k+1},\boldsymbol{t}^{k+1}) - E(\boldsymbol{u}^k,\boldsymbol{t}^k)\right\| \leq \varepsilon$ or $k > N$, jump 7), otherwise, set $k = k+1$, jump 2)
7) Output the solution $\boldsymbol{d}_k$.

3.2. Augmented EFRM Satisfying Karush-Kuhn-Tucker(KKT) Optimal Conditions

Supposed that the magnitude of pixels and edges is bigger than zero, in others words, solutions of (4) are non-negative. Formula (4) is converted into optimal problem with conditioned constraint, which is written as

$$\inf\{E(\boldsymbol{u},\boldsymbol{t})\},\ \boldsymbol{u}\geq 0,\ \boldsymbol{t}\geq 0 \tag{10}$$

According to (10), the first order partial derivative of $\boldsymbol{c}(\boldsymbol{u},\boldsymbol{t}) = [\boldsymbol{u},\boldsymbol{t}]^{\mathrm{T}}$ is written as

$$\nabla \boldsymbol{c} = \left[e_i, e_j\right]^{\mathrm{T}} \tag{11}$$

Where, e_i represents standard ith vector, ∇ represents gradient. From (11), we can know that the constrained conditions of feasible solutions of (10) are independent. If formula (10) has optimal solutions, which must satisfy Karush-Kuhn-Tucker (KKT) conditions [1], which is formulated as

$$\frac{\partial E}{\partial \boldsymbol{d}}\left(\boldsymbol{u}^*, \boldsymbol{t}^*\right) \succ 0 \ , l \succ 0 \tag{12}$$

Where, l represents Lagrangian multiplier. According to complementary conditions of KKT, we have

$$\boldsymbol{d}^{\mathrm{T}} \frac{\partial E}{\partial \boldsymbol{d}}\left(\boldsymbol{u}^*, \boldsymbol{t}^*\right) = 0 \tag{13}$$

According to (13), if $u_j^* = 0$, $t_i^* = 0$,then $\frac{\partial E}{\partial \boldsymbol{d}}\left(\boldsymbol{u}^*, \boldsymbol{t}^*\right) \succ 0$; if $u_i^* > 0$, $t_i^* > 0$,then $\frac{\partial E}{\partial \boldsymbol{d}}\left(\boldsymbol{u}^*, \boldsymbol{t}^*\right) = 0$. In other words, the gradient of (4) may be zero, which makes (7) be zero, we get the optimal solutions. Unfortunately, the computation of large inverse Hessian matrix is costly huge, which leads to the slow convergence speed of Newton iterative algorithm. In order to accelerate the convergence speed of algorithm, the Hessian matrix must be approximated by special structural matrix.

3.3. Structural Hessian Matrix of Newton Iterative Algorithm

In order to make Hessian matrix structure, formula (9) is formulated as structural projection, which is written as

$$\boldsymbol{d}_{k+1} = Proj\left(\boldsymbol{d}_k + \tau_k \boldsymbol{\theta}\right) \tag{14}$$

Where, $\boldsymbol{\theta} = \varLambda_k \bullet \boldsymbol{GE}\left(\boldsymbol{u}_k, \boldsymbol{t}_k\right)$, $\varLambda_k$ represents diagonal matrix, τ_k represents step-length, $Proj\left(\bullet\right)$ is projection operator.

According to (14), the key problem of acceleration of Newton iterative algorithm is how to structure Hessian matrix, and to make characteristic value cluster near 1 and far from zero, which make inverse Hessian matrix compute easily. The preconditioned matrix $\boldsymbol{S}$ is formulated as

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{H}_{12}\boldsymbol{H}_{22}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \lambda \boldsymbol{H}_{22}^{-1}\boldsymbol{H}_{21} & \boldsymbol{I} \end{bmatrix} \tag{15}$$

Where, $\boldsymbol{H}_{11}$ and $\boldsymbol{H}_{22}$ are given by (6). If $\boldsymbol{t}$ is approximated by unitary matrix, and the image is extended by proper periodic boundary conditions, the tight operator $\boldsymbol{A}$ is block circulant circulant block (BCCB) structure, then, $\boldsymbol{H}_{11}$ of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is diagonalized by fast Fourier transform (FFT). Supposed that operator $\boldsymbol{D}$ is approximated by first order differential operator, then, $\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D}$ is diagonalized by FFT, too, that is $\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} = \boldsymbol{F}^*\boldsymbol{\varGamma}_2\boldsymbol{F}$. $\boldsymbol{H}_{11}$ is diagonalized, and $\boldsymbol{H}_{22}$ itself is diagonal matrix, hence, the inverse matrix of $\boldsymbol{S}$ is computed directly. Because of $\boldsymbol{S}^{-1} \bullet \boldsymbol{S} = \boldsymbol{I}$, the search direction of (7) is reformulated as

$$\boldsymbol{HE}\left(\boldsymbol{u}, \boldsymbol{t}\right) \bullet \boldsymbol{S}^{-1} \bullet \boldsymbol{S}\theta = -\boldsymbol{GE}\left(\boldsymbol{u}, \boldsymbol{t}\right) \tag{16}$$

Formula (16) is simplified as

$$\boldsymbol{HE}\left(\boldsymbol{u}, \boldsymbol{t}\right) \bullet \boldsymbol{S}^{-1} = \begin{bmatrix} \boldsymbol{B} & \left(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{H}_{11}^{-1}\right)\boldsymbol{H}_{12}\boldsymbol{H}_{22}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{17}$$

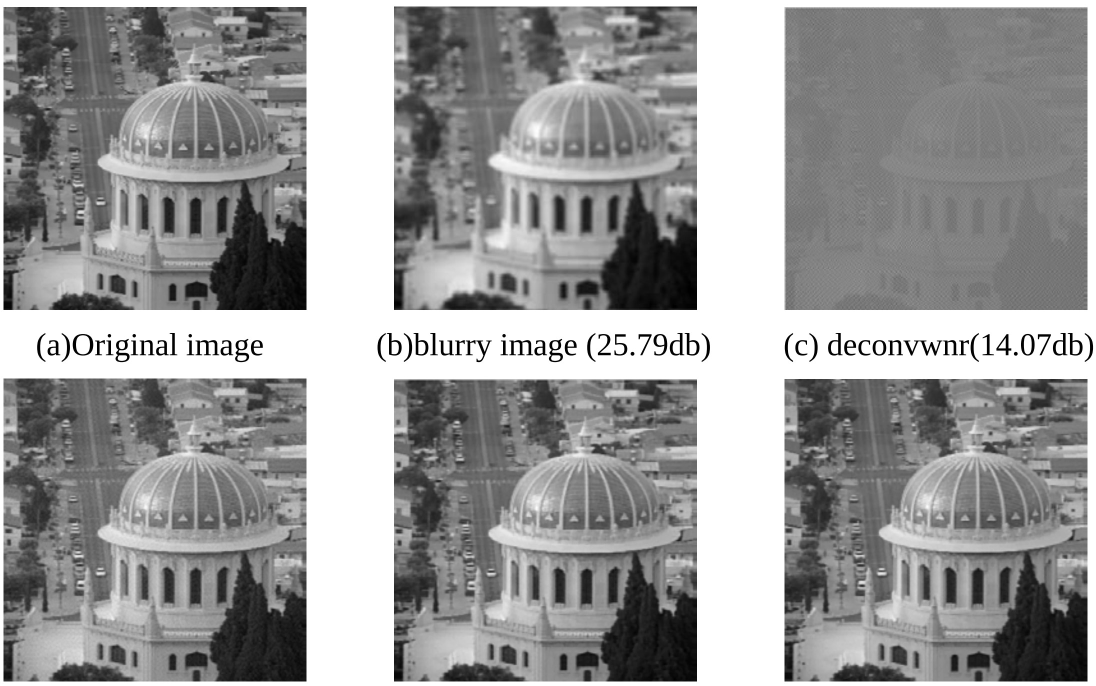
Where $\boldsymbol{B} = \left(\boldsymbol{H}_{11} - \boldsymbol{H}_{12}\boldsymbol{H}_{22}^{-1}\boldsymbol{H}_{21}\right)\boldsymbol{H}_{11}^{-1}$. Formula (17) is inverted and upper triangle block matrix, the characteristic values are clustered near 1. Set $\boldsymbol{\theta} = \boldsymbol{S}\theta$, the search direction is obtained.

3.4. Structural Newton Iterative Algorithm

1) Initialization, set $k = 0$, $\boldsymbol{u}^0$, $\boldsymbol{t}^0$, maximum iterative number N.
2) According to (5), (6), compute gradient and Hessian matrix of EFRM. Compute structural matrix by (15).
3) Using conjugate gradient algorithm for computing (16), then get the search direction of Newton iterative algorithm; Using line back search algorithm, then update step length, and update the solutions of EFRM by (14).
4) If $\left\| E\left(\boldsymbol{u}^{k+1},\boldsymbol{t}^{k+1}\right) - E\left(\boldsymbol{u}^{k},\boldsymbol{t}^{k}\right) \right\| \le \varepsilon$ or $k > N$, jump 5); otherwise, set $k = k + 1$, jump 2).
5) Output restoration image $\boldsymbol{u}^*$.

4. Experiment Results and Analysis

In this section, we illustrate the effectiveness of the proposed method, which is compared with deconvwnr function, deconvreg function, functions can be obtained from Matlab toolbox. Deconvwnr function restores blurry image using the Wiener filter algorithm, which is optimal in a sense of the fitting term described by L_2 norm. Deconvreg function restores blurry image using the regularized filter algorithm, which is optimal in the sense of fitting term described by L_2 norm and regularized term described by the Laplacian operator. FISTA restores blurry image using first order method, which is optimal in the sense of fitting term characterized by L_2 norm and regularized term characterized by TV semi-norm. Our tests are done by using Matlab 7.12.0 (R2011a) with Intel(R) Core(TM) i7-4700MQ CPU 2.20 GHz and 8GB memory.



(a)Original image (b)blurry image (25.79db) (c) deconvwnr(14.07db)

(d) deconvreg(33.29db) (e) FISTA (34.33db) (f) the proposed (35.62db)

Figure 1 Restoration comparison of different algorithms. .

In figure 1, figure 1(a) is original image, figure 1(b) is blurred by imaging system and Gaussian noise, and a lot of edges are lost and PSNR is 25.79db. Figure 1(c) is restored by deconvwer function that belongs to Wiener filter, restoration result is worse, the edges of image is completely lost. Deconvwer function only takes advantage of fitting term and tries to get solution by computing invert matrix of imaging system. However, imaging system is ill-posed. PSNR of figure 1(c) is 14.07db, it is lower than PSNR of blurry image. Figure 1(d) is restored by deconvreg function, the edge of restoration is smooth, and PSNR of figure 1(d) is bigger than that of figure 1(b). Figure 1(e) is produced by FISTA, the edge of restoration visual effect is better than that of figure 1(d), and PSNR of figure 1(e) is higher than that of figure 1(d). Figure 1(f) is restored by the proposed algorithm, PSNR is higher than that of other algorithms, which shows that the proposed algorithm is better than other algorithms.

5. Conclusions

Fitting term is described by L_2 norm, regularization term is characterized by half quadratic function, by introducing auxiliary variable, the primal EFRM is converted into augmented EFRM. Hessian matrix of augmented EFRM is determined by fitting term and regularization term, because of Hessian matrix without special structure, it is difficult to compute the inverse Hessian matrix. In order to solve the problem, by imposing periodic boundary condition on image and constructing preconditioned matrix, Hessian matrix is converted into diagonal matrix. Based on classical Newton iterative algorithm, a projection Newton iterative algorithm based on preconditioned matrix is proposed. Comparing with other state-of-art algorithms, the proposed method obtains better visual effect and higher PSNR.

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