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Local Bifurcation of Steady State Solutions for A Class of Reaction-Diffusion System

Yupei Zhang[1],Xibing He[2]

[1]Nanchang Institute of Science and Technology, Nanchang,330108, China

[2]Department of Mathematics and Computer Science, Nanchang Normal University, Nanchang,330032, China

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Abstract. In this paper,we study local bifurcation fromthe eigenvalue $\lambda = \lambda_9$ with multiplicity two of the Laplacian operator for the steady-state solutions of a class of reaction-diffusion equation with Robin boundary conditions on the two-dimensional rectangular area $[0, 2\pi] \times [0, \pi]$.

Introduction

In bifurcation theory,a natural problem is whether accurate descriptions parallel to that in Crandall-Rabinowitz[1][2]theorem are still possible at eigenvalueswith multiplicity greater than one,at least in special cases[3][5].Concerning eigenvalues of higher multiplicity, [4] are known of potential operators where bifurcation takes place.Local bifurcation from the branch of trivial solutions in an equation of the form $F(\lambda, u) = \Delta u + \lambda u + f(x, u) = 0$ where $\lambda \in R^1, \Omega \subset R^n$ is a bounded domain and $f(x, u) = o(u)$ as $u \to 0$ has been widely treated in the literature.

Preliminaries

In this paper,we restrict ourselves in what follows to a special case of the reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \lambda u + f(\bar{x}, u) & in\ \bar{x} = (x, y) \in \Omega \\ u(0, y, t) = u(2\pi, y, t) = 0 & \forall y \in [0, \pi], t > 0 \\ \dfrac{\partial u}{\partial n}(x, 0, t) = \dfrac{\partial u}{\partial n}(x, \pi, t) = 0 & \forall x \in [0, 2\pi], t > 0 \\ u(\bar{x}, 0) = u_0(\bar{x}) & t = 0 \end{cases} \quad (1)$$

where $t \in [0, +\infty), \lambda \in R^1$ and f satisfies the two following conditions:(i) $f \in C^3(\bar{\Omega} \times R^1)$; (ii) $f(\bar{x}, 0) = f_u(\bar{x}, 0) = f_{uu}(\bar{x}, 0) = 0,\ f_{uuu}(\bar{x}, 0) = k \neq 0.$

It is easy to find that the question of steady state bifurcation from the double eigenvalue in(1) can be converted into the bifurcation problem of the following semi-linear elliptic equation:

$$\begin{cases} \Delta u + \lambda u + f(\bar{x}, u) = 0 & in\ \bar{x} = (x, y) \in \Omega \\ u(0, y) = u(2\pi, y) = 0 & \forall y \in [0, \pi] \\ \dfrac{\partial u}{\partial n}(x, 0) = \dfrac{\partial u}{\partial n}(x, \pi) = 0 & \forall x \in [0, 2\pi] \end{cases} \quad (2)$$

Next, we carry on the Taylor expansion about f at the point of $u = 0$ and we can get that

$$f(\bar{x}, u) = f(\bar{x}, 0) + f_u(\bar{x}, 0)u + \frac{1}{2!} f_{uu}(\bar{x}, 0)u^2 + \frac{1}{3!} f_{uuu}(\bar{x}, 0)u^3 + o(u^3) = \frac{k}{6}u^3 + o(u^3) = u^3\left(\frac{k}{6} + o(1)\right)$$

Thus if $k>0$,we can consider the case $f(\bar{x},u)=u^3(1+\theta(\bar{x},u))$ where $\theta\in C^1(\bar{\Omega}\times R^1)$ and $\theta(\bar{x},0)=0$ .Of course,if $k<0$,accordingly we consider $f(\bar{x},u)=-u^3(1+\theta(\bar{x},u))$.

The Main Results

Theorem 1 Let $\Omega=[0,2\pi]\times[0,\pi]\subset R^2$ .Then there exist an $\varepsilon>0$ and a neighbourhood U of $(\lambda_9,0)$ in $R^1\times C(\bar{\Omega})$ such that the set of all steady-state bifurcation solutions of (1) in U can be described as the union of four C^1 curves: $s\in(-\varepsilon,\varepsilon)\mapsto(\lambda_i(s),u_i(s))$, $i=1,\cdots,4$, such that

$$\begin{cases}\lambda_i(s)=\lambda_9+\sigma_i s^2+o(s^2)\\ u_i(s)=s\phi_{\alpha_i}+o(s)\end{cases}$$

where $\alpha_1=0,\alpha_2=\pi/4,\alpha_3=\pi/2,\alpha_4=3\pi/4$, and

$\sigma_1=\sigma_3=-9/16,\sigma_2=\sigma_4=-21/32$, for $k>0$

$\sigma_1=\sigma_3=9/16,\sigma_2=\sigma_4=21/32$, for $k<0$.

Proof.On the basis of calculation,we label M as the vector space of all eigenfunctions $A=\sin x\cos 2y$ and $B=\sin 2x\cos y$ associated to the double eigenvalue $\lambda_9=\lambda_{2,2}=\lambda_{1,4}=5$ and denote $M^{\perp}=\left\{u\in C(\bar{\Omega}):\int_{\Omega}u\phi=0,\forall\phi\in M\right\}$. After that we introduce the normalizedeigenfunction $\phi_\alpha(x,y)=\cos\alpha\sin x\cos 2y+\sin\alpha\sin 2x\cos y=\cos\alpha A+\sin\alpha B,\alpha\in[0,2\pi)$,which is a parametric representation of all eigenfunctions ϕ with $\|\phi\|_{L^2}=\pi/\sqrt{2}$. Alongside with ϕ_α weintroduce an orthogonal eigenfunction defined by $\psi_\alpha(x,y)=\sin\alpha\sin x\cos 2y-\cos\alpha\sin 2x\cos y$. Notice that $\phi_{\alpha-\pi/2}=\psi_\alpha$ and $D_\alpha\phi_\alpha=-\psi_\alpha$. Let (λ_n,u_n) be a sequence of solutions to (2) such that $\lambda_n\to\lambda_9$ and $u_n\to 0$ in $C(\bar{\Omega})$.We make the normalization: $\tilde{u}_n=u_n/\|u_n\|_\infty$. Then $\tilde{u}_n$ verifies the equation

$$\begin{cases}\Delta\tilde{u}_n+\lambda_n\tilde{u}_n\pm\tilde{u}_n u_n^2(1+\theta(\bar{x},u))=0 & in\ \Omega\\ \tilde{u}_n(0,y)=\tilde{u}_n(2\pi,y)=0 & \forall y\in[0,\pi]\\ \dfrac{\partial\tilde{u}_n}{\partial n}(x,0)=\dfrac{\partial\tilde{u}_n}{\partial n}(x,\pi)=0 & \forall x\in[0,2\pi]\end{cases}\quad(3)$$

The first formula of (3) can be transformed into $(-\Delta)^{-1}\left[\lambda_n\pm u_n^2(1+\theta(\bar{x},u))\right]\tilde{u}_n=\tilde{u}_n$, Thus that (3) is equivalent to a fixed point equation for a self-sequential compact operator in $C(\bar{\Omega})$. We all know that a compact operator can map a bounded set into a compact set,and taking into account that $\|\tilde{u}_n\|_\infty=1$,passing to a subsequence still denoted by $\tilde{u}_n$,we have that $\tilde{u}_n\to u_0$ in $C(\bar{\Omega})$ with $\|u_0\|_\infty=1$ and u_0 satisfies

$$\begin{cases}\Delta u_0+\lambda_9 u_0=0 & in\ \Omega\\ u_0(0,y)=u_0(2\pi,y)=0 & \forall y\in[0,\pi].\\ \dfrac{\partial u_0}{\partial n}(x,0)=\dfrac{\partial u_0}{\partial n}(x,\pi)=0 & \forall x\in[0,2\pi]\end{cases}$$

It follows that $u_0\in M$ and for some $\alpha\in[0,2\pi)$ we can assume $u_0=c\phi_\alpha$ with $c=\|\phi_\alpha\|^{-1}$ .Writing $\tilde{u}_n=\phi_n+\psi_n$ with $\phi_n\in M,\psi_n\in M^{\perp}$ (so that $\phi_n\to c\phi_\alpha,\psi_n\to 0$);again writing

$t_n = \|u_n\|_\infty$ and simplifying the first formula of(3),we arrive that

$$\Delta(\phi_n+\psi_n)+\lambda_9(\phi_n+\psi_n)+(\lambda_n-\lambda_9)(\phi_n+\psi_n)\pm t_n^2(\phi_n+\psi_n)^3\left(1+\theta\left(\bar{x},t_n(\phi_n+\psi_n)\right)\right)=0.$$

Since $\phi_n \in M$, we get that $\Delta\phi_n+\lambda_9\phi_n=0$ .Then the above mathematical expression can be converted to

$$\Delta\psi_n+\lambda_9\psi_n+(\lambda_n-\lambda_9)(\phi_n+\psi_n)\pm t_n^2(\phi_n+\psi_n)^3\left(1+\theta\left(\bar{x},t_n(\phi_n+\psi_n)\right)\right)=0 \quad (4)$$

Multiplying by ϕ_α and integrating by parts we obtain $\frac{(\lambda_n-\lambda_9)}{t_n^2}\int_\Omega \phi_n\phi_\alpha \pm \int_\Omega(\phi_n+\psi_n)^3\left(1+\theta\left(\bar{x},t_n(\phi_n+\psi_n)\right)\right)\phi_\alpha=0$ from which it follows after passing to the limit that $\lim_{n\to\infty}\frac{\lambda_n-\lambda_9}{t_n^2}=\mp c^2\frac{\int_\Omega \phi_\alpha{}^4}{\int_\Omega \phi_\alpha{}^2}$. Similarly,multiplying (4)by ψ_α, integrating by parts and passing to the limit we get that $\frac{\int_\Omega \phi_\alpha{}^4}{\int_\Omega \phi_\alpha{}^2}\int_\Omega \phi_\alpha\psi_\alpha=\int_\Omega \phi_\alpha{}^3\psi_\alpha$. We have the left hand equals 0 from $\int_\Omega \phi_\alpha\psi_\alpha=\int_0^{2\pi}dx\int_0^{\pi}(\cos\alpha A+\sin\alpha B)(\sin\alpha A-\cos\alpha B)dy=0$. For the right hand,it can be checked that $\int_\Omega \phi_\alpha{}^3\psi_\alpha=\int_0^{2\pi}dx\int_0^{\pi}(\cos\alpha A+\sin\alpha B)^3(\sin\alpha A-\cos\alpha B)dy=-\frac{3}{64}\pi^2\sin 2\alpha\cos 2\alpha$,so we obtain that the bifurcation is only possible from four values of α ,namely $\alpha=0,\pi/4,\pi/2,3\pi/4$ .Notice that there are other values of α .Since $\phi_{\pi+\alpha}=-\phi_\alpha$,we can find that the bifurcation occurs only at $\alpha=0,\pi/4,\pi/2,3\pi/4$. Writing $s=ct_n,\sigma=\mp\frac{\int_\Omega \phi_\alpha{}^4}{\int_\Omega \phi_\alpha{}^2}$,we have $\lim_{n\to\infty}\frac{\lambda_n-\lambda_9}{c^2t_n^2}=\lim_{n\to\infty}\frac{\lambda_n-\lambda_9}{s^2}=\sigma$,that is $\lambda_n=\lambda_9+s^2\sigma+o(s^2)$ .Through calculation we can get $\sigma_1=\sigma_3=-9/16,\sigma_2=\sigma_4=-21/32$ for $k>0$. and $\sigma_1=\sigma_3=9/16,\sigma_2=\sigma_4=21/32$ for $k<0$. Besides if followsfrom(4)that $\Delta\psi_n+\lambda_9\psi_n=\mathrm{O}(t_n^2)$. Since $(\Delta+\lambda_9)^{-1}$ is a bounded linear operator from $M^\perp$ into itself,we can get

$$\psi_n=\mathrm{O}(t_n^2),u_n=\|u_n\|_\infty\tilde{u}_n=t_n\left(c\phi_{\alpha+o(1)}+\mathrm{O}(t_n^2)\right)=s\phi_{\alpha+o(1)}+\mathrm{O}(s^3).$$

As a consequence of this analysis,any solution (λ_n,u_n) near the bifurcation point $(\lambda_9,0)$ has the form $\begin{cases}\lambda_n=\lambda_9+s^2\sigma+o(s^2)\\ u_n=s\phi_{\alpha+o(1)}+\mathrm{O}(s^3)\end{cases}$ .Nowwe turn to the actual construction of the bifurcated branches.Let α_0 be fixed as one of the four values $\alpha=0,\pi/4,\pi/2,3\pi/4$ given above.For s small we want to solve it: $\begin{cases}\Delta\psi+\lambda_9\psi+\sigma\phi_\alpha+s^2\sigma\psi\pm\left(\phi_\alpha+s^2\psi\right)^3\left(1+\theta\left(\bar{x},s\phi_\alpha+s^3\psi\right)\right)=0 & in\ \Omega\\ \psi(0,y)=\psi(2\pi,y)=0 & \forall y\in[0,\pi]\\ \frac{\partial\psi}{\partial n}(x,0)=\frac{\partial\psi}{\partial n}(x,\pi)=0 & \forall x\in[0,2\pi]\end{cases}$. Denoting by K the inverse of Δ which is a compact and linear operator from $C(\bar{\Omega})$ into itself.

For α,σ,ψ in a small neighbourhood of α_0,σ_0,ψ_0 respectively,the above problem is equivalent to $H(\alpha,\sigma,\psi,s)=0$ where

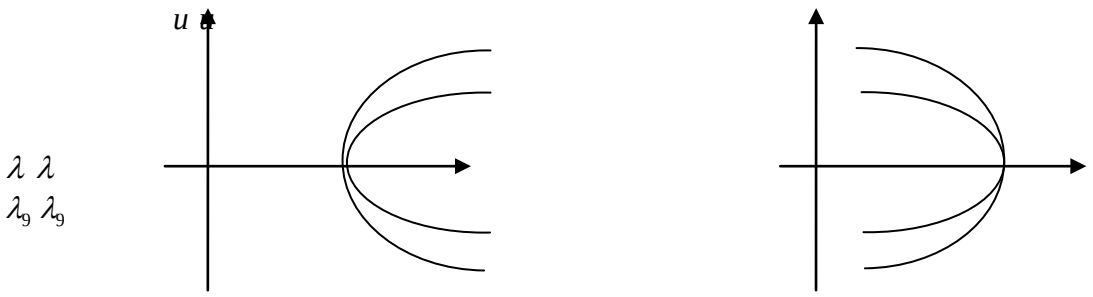
$H(\alpha,\sigma,\psi,s)=\psi+K\left(\lambda_9\psi+\sigma\phi_\alpha+s^2\sigma\psi\pm\left(\phi_\alpha+s^2\psi\right)^3\left(1+\theta\left(\overline{x},s\phi_\alpha+s^3\psi\right)\right)\right)$ and $\psi_0\in M^\perp$ isthe unique solution of this equation

$$\begin{cases}\Delta\psi+\lambda_9\psi+\sigma_0\phi_{\alpha_0}\pm\phi_{\alpha_0}{}^3=0 & in\ \Omega\\ \psi(0,y)=\psi(2\pi,y)=0 & \forall y\in[0,\pi].\\ \dfrac{\partial\psi}{\partial n}(x,0)=\dfrac{\partial\psi}{\partial n}(x,\pi)=0 & \forall x\in[0,2\pi]\end{cases}$$

Let us apply the implicit function theorem in our setting.First we must notice that H is a C^1 function ofits arguments in a neighbourhood Q of $(\alpha_0,\sigma_0,\psi_0,0)$ in $R^1\times R^1\times M^\perp\times R^1$. Also, $H(\alpha_0,\sigma_0,\psi_0,0)=0$ and $D_{(\alpha,\sigma,\psi)}H(\alpha_0,\sigma_0,\psi_0,0)(\tilde\alpha,\tilde\sigma,\tilde\psi)=\tilde\psi+K\left(\lambda_9\tilde\psi+\tilde\sigma\phi_{\alpha_0}-\tilde\alpha\left(\sigma_0\psi_{\alpha_0}\pm3\phi_{\alpha_0}^2\psi_{\alpha_0}\right)\right)$ For some $(\tilde\alpha,\tilde\sigma,\tilde\psi)\in R^1\times R^1\times M^\perp$, we assume that $D_{(\alpha,\sigma,\psi)}H(\alpha_0,\sigma_0,\psi_0,0)(\tilde\alpha,\tilde\sigma,\tilde\psi)=0.$ This means that $\tilde\psi$ solve the following problem

$$\begin{cases}\Delta\tilde\psi+\lambda_9\tilde\psi+\tilde\sigma\phi_{\alpha_0}-\left(\sigma_0\psi_{\alpha_0}\pm3\phi_{\alpha_0}{}^2\psi_{\alpha_0}\right)\tilde\alpha=0 & in\,\Omega\\ \tilde\psi(0,y)=\tilde\psi(2\pi,y)=0 & \forall y\in[0,\pi].\ (5)\\ \dfrac{\partial\tilde\psi}{\partial n}(x,0)=\dfrac{\partial\tilde\psi}{\partial n}(x,\pi)=0 & \forall x\in[0,2\pi]\end{cases}$$

Multiplying by ϕ_{α_0} ,integrating in Ω and performing an integration by parts,we arrive at $\tilde\sigma=0$ since $\int_\Omega\phi_\alpha{}^3\psi_\alpha=0$ holds.Multiplying by ψ_{α_0} instead,we get $\left(\sigma_0\int_\Omega\psi_{\alpha_0}{}^2\pm3\int_\Omega\phi_{\alpha_0}^2\psi_{\alpha_0}^2\right)\tilde\alpha=0.$ In view of that $\int_\Omega\psi_{\alpha_0}^2=\dfrac{\pi^2}{2}$ and $\int_\Omega\phi_{\alpha_0}^2\psi_{\alpha_0}^2=\dfrac{\pi^2}{8}-\dfrac{3}{64}\pi^2\sin^2 2\alpha_0$,it is easy to see that the term inside brackets is always nonzero.Thus $\tilde\alpha=0$ and (5)leads to $\tilde\psi=0.$ Tosummarize, $D_{(\alpha,\sigma,\psi)}H(\alpha_0,\sigma_0,\psi_0,0)(\tilde\alpha,\tilde\sigma,\tilde\psi)$ is one-to-one and hence an isomorphism since it can be viewed as a compact perturbation of the identity. $\forall\varepsilon>0,$ the implicit function theorem applies to three C^1 functions $\alpha:(-\varepsilon,\varepsilon)\to R^1$, $\sigma:(-\varepsilon,\varepsilon)\to R^1,\psi:(-\varepsilon,\varepsilon)\to M^\perp$ such that $\alpha(0)=\alpha_0$, $\sigma(0)=\sigma_0$, $\psi(0)=\psi_0$ and the set of solutions of $H(\alpha,\sigma,\psi,s)=0$ near the point $(\alpha_0,\sigma_0,\psi_0,0)$ can be expressed as $(\alpha(s),\sigma(s),\psi(s),s)$ This conclusion together with the form of (λ_n,u_n) gives in particular a unique curve of solutions to(2)such that $u(s)\sim s\phi_{\alpha_0}$ as $s\to0.$ Since α_0 can be taken as any of the four values $0,\pi/4,\pi/2,3\pi/4.$,we have exactly four branches of solutions near the bifurcation point $(\lambda_9,0)$ .The proof of the theorem is thus complete and the bifurcation graphic looks like this:



(i)for $k<0$ (ii)for $k>0$

Conclusion

Based on the analysis,we should mention that the nonlinearity $\pm u^3\left(1+\theta\left(\overline{x},u\right)\right)$ can be replaced by $\pm u^{2m+1}\left(1+\theta\left(\overline{x},u\right)\right), m\in N^+$, with no change basically in the proofs above together with mathematical induction. The next step of our work is that case $f\left(\overline{x},u\right)=\pm u^2\left(1+\theta\left(\overline{x},u\right)\right)$ and the corresponding local and steady state bifurcation problem of (1).

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