

Nonlinear biharmonic equation with Hardy-Sobolev potential

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Abstract: This paper deals with the Dirichlet problem of a superlinear biharmonic equation with Hardy-Sobolev potential, where the nonlinearity is odd. By Fountain theorem with Cerami condition and Sobolev-Hardy inequality, we get the existence of nontrivial solutions for the problem.

1 Introduction

In this paper, we will consider the existence of multiple solutions of biharmonic equation with Dirichlet boundary Problem:

$$\begin{cases} \Delta^2 u = \lambda \frac{u}{|x|^s} + f(x, u), & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a smooth bounded domain in \mathbb{R}^N ($N > 4$) and $0 \in \Omega$, Δ^2 denotes the biharmonic operator, $\lambda > 0$ is a parameter, $1 \leq s \leq 4$, $\frac{u}{|x|^s}$ is called Hardy-Sobolev potential.

In recent years, the problems with singularity have been a hot topic. But because of singularity, the problems are also difficult. In problem (1.1), if $f(x, u) = f(x)$ or $f(x, u) = |u|^{p-2}u$, $2 < p \leq \frac{2N}{N-4}$ when $0 \leq s \leq 2$, Kang and Deng [5] studied the existence of the solutions of the problem (1.1) by using variational methods and Sobolev-Hardy inequalities. If $f(x, u) = |u|^{\frac{N+4}{N-4}}u$ in problem (1.1), by means of the concentrate compactness principle [6] and Mountain Pass theorem, Xiong and Shen [11] got the two existence results for the problem (1.1). Meanwhile, the authors proved, by Pohozaev identity for singular solution, there is no nontrivial solution for equations with critical exponent and critical potential [2, 3, 8]. If $f(x, u) = f(x) + \frac{|u|^{2^*(\alpha)-2}u}{|x|^\alpha}$, $2^*(\alpha) = \frac{2(N-\alpha)}{N-4}$, $0 \leq \alpha < 4$ and $s = 4$ in problem (1.1), Hu and Song [4] got the existence of at least one solution under some certain condition by employing variational method and Sobolev inequality.

In this paper, our main interests are the problem (1.1) suggested by Kang and Deng [5], Liu and Li [7], Wang and Shen [9] and Xiong and Shen [11]. Our main methods follow that of Liu and Li [7] and Wang and Shen [9].

Throughout this paper, we denote the norms of u in $H_0^2(\Omega)$ and $L^p(\Omega)$ by $\|u\| = (\int_{\Omega} |\Delta u|^2 dx)^{1/2}$ and $\|u\|_p = (\int_{\Omega} |\Delta u|^p dx)^{1/p}$, respectively. Here, $H_0^2(\Omega)$ is the Sobolev space $W_0^{2,2}(\Omega)$ with respect the norm $\|u\|$. \rightharpoonup for weak convergence and \rightarrow for strong convergence.

$u \in H_0^2(\Omega)$ is said to be a weak solution of the problem (1.1) if u satisfies

$$\int_{\Omega} \Delta u \Delta \varphi dx - \lambda \int_{\Omega} \frac{u}{|x|^s} \varphi dx - \int_{\Omega} f(x, u) \varphi dx = 0, \forall \varphi \in H_0^2(\Omega).$$

It is well known that the nontrivial solutions of the problem (1.1) are equivalent to the nonzero critical points of the energy functional

$$I_{\lambda}(u) = \frac{1}{2} \int_{\Omega} |\Delta u|^2 dx - \frac{\lambda}{2} \int_{\Omega} \frac{u^2}{|x|^s} dx - \int_{\Omega} F(x, u) dx, \quad (1.2)$$

for $u \in H_0^2(\Omega)$, where $F(x, u) = \int_0^u f(x, s) ds$. So the Frechet derivative $I'(u)$ of the energy functional (1.2) at u is

$$\langle I'(u), \varphi \rangle = \int_{\Omega} \Delta u \Delta \varphi dx - \lambda \int_{\Omega} \frac{u}{|x|^s} \varphi dx - \int_{\Omega} f(x, u) \varphi dx, \forall \varphi \in H_0^2(\Omega).$$

Let $\Phi(t, s) = \frac{1}{2} t^{\alpha} f(x, s) s - F(x, ts)$, $\alpha \in (0, 2)$ is a constant. Assume that function $f(x, s)$ satisfies

(f₁) $f(x, s) \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x, 0) = 0$ for any $x \in \Omega$;

(f₂) there exists some constants $a, b > 0$ such that $|f(x, s)| \leq a + b|s|^{q-1}$ for $(x, s) \in \Omega \times \mathbb{R}$, where $2 < q < \frac{2N}{N-4}$;

(f₃) $\lim_{|s| \rightarrow \infty} \frac{f(x, s)s}{|s|^2} = \infty$ uniformly $x \in \Omega$;

(f₄) there exist $\mu > 0$ and $\delta > 0$ such that $\Phi(t, s) \leq \mu \Phi(1, s)$, for any $x \in \Omega$, $|s| \geq \delta$ and $t \in (0, 1]$;

(f₅) $f(x, -s) = -f(x, s)$, $\forall (x, s) \in \Omega \times \mathbb{R}$;

Assume $0 \leq s \leq 4$, then there exists a constant $C > 0$ such that

$$\left(\int_{\Omega} \frac{|u|^q}{|x|^s} dx \right)^{\frac{1}{q}} \leq C \left(\int_{\Omega} |\Delta u|^2 dx \right)^{\frac{1}{2}}, \forall u \in H_0^2(\Omega).$$

We use $\lambda_s(\Omega)$ to denote the best Sobolev-Hardy constant, that is, the largest constant C satisfying the above inequality for all $u \in H_0^2(\Omega)$, i.e.,

$$\lambda_s(\Omega) = \inf_{u \in H_0^2(\Omega), |u| \neq 0} \frac{\int_{\Omega} |\Delta u_k|^2 dx}{\int_{\Omega} \frac{|u|^2}{|x|^s} dx}$$

The main result of this paper is stated as follows.

Theorem 1.1 *Let (f₁)-(f₅) hold. If $\lambda < \lambda_s(\Omega)$, then problem (1.1) has infinitely many nontrivial solutions $\{u_k\}$ in $H_0^2(\Omega)$ which satisfies that*

$$\frac{1}{2} \int_{\Omega} |\Delta u_k|^2 dx - \frac{\lambda}{2} \int_{\Omega} \frac{u_k^2}{|x|^s} dx - \int_{\Omega} F(x, u_k) dx \rightarrow +\infty \quad \text{as } k \rightarrow +\infty.$$

2 Preliminaries

Throughout this paper, we let E be a real Banach space endowed with the norm $\|\cdot\|$ and E^* the topological dual of E . For all $x \in E$ and $x^* \in E^*$, the value of f at x is denoted by $\langle x, f \rangle$ and is called the duality pairing. Then, we give the main definitions and lemmas in this section.

Definition 2.1 [10] Let E be a real Banach space, a functional $I \in C^1(E, \mathbb{R})$ is said to satisfy the Palais-Smale condition (PS)-condition, for short if there exists a constant $c \in \mathbb{R}$, any sequence $\{u_n\} \subset E$ satisfying

$$I(u_n) \rightarrow c, I'(u_n) \rightarrow 0, \text{ as } n \rightarrow \infty, \quad (2.1)$$

possesses a convergent subsequence.

Definition 2.2 [7, 9] A functional $I \in C^1(E, \mathbb{R})$, where E be a real Banach space, is said to satisfy the Cerami-condition ((C)-condition, for short) in (c_1, c_2) ($-\infty \leq c_1 < c_2 \leq +\infty$) if for any $c \in (c_1, c_2)$,

- (i) any bounded sequence $\{u_n\} \subset E$ satisfying (2.1) possesses a convergent subsequence, and
- (ii) there exist $\delta, M, \epsilon > 0$ such that $\|I'(u_n)\|_{E^*} \|u_n\| \geq \epsilon$ for any $u \in I^{-1}[c - \delta, c + \delta]$ with $\|u_n\| \geq M$.

It is obvious that (C)-condition is slightly weaker than (PS)-condition while the most important implications of (PS)-condition are retained.

Let E is a separable Banach space, and there exist $\{e_n\}_{n \in \mathbb{N}} \subset E$ and $\{\phi_n\}_{n \in \mathbb{N}} \subset E^*$ such that

- (i) $\langle \phi_n, e_m \rangle = \delta_n^m$, where $\delta_n^m = 1$ if $n = m$, $\delta_n^m = 0$ if $n \neq m$, and
- (ii) $\overline{\text{span}}\{e_n, n \in \mathbb{N}\} = E$, $\overline{\text{span}^{w^*}}\{\phi_n, n \in \mathbb{N}\} = E^*$.

Let $E_j = \text{span}\{e_j\}$, then $E = \overline{\bigoplus_{j \geq 1} E_j}$. Set $Y_k = \overline{\bigoplus_{j=1}^k E_j}$, $Z_k = \overline{\bigoplus_{j \geq k} E_j}$. Thus, we have the following lemma:

Lemma 2.1 [1, 7, 10](Fountain theorem) Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ with I even, that is, $I(-u) = I(u)$. Suppose $I(0) = 0$ and I satisfies (C)-condition, and for any $k \in \mathbb{N}$, there exists $\rho_k > r_k > 0$ such that

- (i) $b_k = \inf_{u \in Z_k, \|u\|=r_k} I(u) \rightarrow +\infty$, as $k \rightarrow +\infty$, and
- (ii) $a_k = \max_{u \in Y_k, \|u\|=\rho_k} I(u) \leq 0$,

then I possesses an unbounded sequence of critical values.

Lemma 2.2 [12] Assume $1 \leq q \leq 2^*(s) = \frac{2(N-s)}{N-4}$, then

(1) (Sobolev-Hardy inequality) for any $u \in H_0^2(\Omega)$, there exists a constant C such that

$$\left(\int_{\Omega} \frac{|u|^q}{|x|^s} dx \right)^{\frac{1}{q}} \leq C \left(\int_{\Omega} |\Delta u|^2 dx \right)^{\frac{1}{2}};$$

(2) the mapping $u \rightarrow \frac{u}{|x|^{s/q}}$ from $H_0^2(\Omega)$ into $L^q(\Omega)$ is compact provided $1 \leq q \leq 2^*(s)$.

Lemma 2.3 Assume (f_1) - (f_4) hold, and suppose that there exists $\{u_k\} \subset H_0^2(\Omega)$ and $\{t_k\} \subset \mathbb{R}, t_k > 0$ such that $\langle I'(u_k), u_k \rangle \rightarrow 0$ and $t_k \rightarrow 0$ as $k \rightarrow \infty$. Let $w_k = t_k u_k$, then $I(w_k) \leq \mu I(u_k) + o(1)$ as $k \rightarrow \infty$.

Proof From the assumption of this lemma, it is easy to know that $t_k^2 \langle I'(u_k), u_k \rangle \rightarrow 0$ as $k \rightarrow \infty$, and consequently, we have

$$\|w_k\|^2 = \lambda \int_{\Omega} t_k^2 \frac{u_k^2}{|x|^s} dx - \int_{\Omega} t_k^2 f(x, u_k) u_k dx + o(1).$$

Hence, from (f_4) , we have

$$I(w_k) = \frac{1}{2} \int_{\Omega} t_k^2 f(x, u_k) u_k dx - \int_{\Omega} F(x, u_k) dx + o(1) \leq \mu \int_{\Omega} \Phi(x, u_k) dx + o(1) = \mu I(u_k) + o(1).$$

Lemma 2.4 Assume that (f_1) - (f_4) hold, if $\lambda < \lambda_s(\Omega)$, then functional I satisfies (C)-condition.

Proof By Sobolev embedding theorem, we know that $H_0^2(\Omega) \rightarrow L^q(\Omega), 1 \leq q < 2^* = \frac{2N}{N-4}$ is compact. Meanwhile, from Lemma 2.2, it is easy to show that I satisfies (C)-condition (i).

Next, we prove that I satisfies (C)-condition (ii). Assume that there exists a constant $c \in \mathbb{R}$ and for any sequence $\{u_k\} \subset E$ such that

$$I(u_k) \rightarrow c, \|u_k\| \rightarrow \infty, \|I'(u_k)\|_{E^*} \|u_k\| \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (2.2)$$

Obviously, from the assumptions above, we obtain that $\langle I'(u_n), u_n \rangle \rightarrow 0$. For any constant $K > 0$, set

$$t_k = \frac{\sqrt{2K}}{\|u_k\|}, w_k = t_k u_k = \frac{\sqrt{2K} u_k}{\|u_k\|}, \quad (2.3)$$

It is from (2.3) that $\{w_k\}$ is bounded in $H_0^2(\Omega)$, and hence there exists $w \in H_0^2(\Omega)$ such that $w_k \rightharpoonup w$ in $H_0^2(\Omega)$; $w_k \rightarrow w$ in $L^q(\Omega), 1 \leq q < 2^*$; $w_k \rightarrow w, \text{ a.e. } x \in \Omega$.

We will show the contradiction for the following two cases:

Case a) Suppose $w \equiv 0$, then from $w_k \rightarrow w, \text{ a.e. } x \in \Omega$, we have

$$\int_{\Omega} F(x, w_k) dx \rightarrow 0, k \rightarrow \infty.$$

Thus, by Sobolev-Hardy inequality, we have

$$I(w_k) \geq \frac{1}{2} \|w_k\|^2 - \frac{\lambda}{2} \int_{\Omega} \frac{|w_k|^2}{|x|^s} dx - C \geq \frac{\sigma}{2} \|w_k\|^2 - C.$$

where $\sigma = \frac{\lambda_s(\Omega) - \lambda}{\lambda_s(\Omega)}$.

By Lemma 2.3, we have

$$\|w_k\|^2 \leq \frac{2}{\sigma} I(w_k) + C \leq \frac{2\mu}{\sigma} I(w_k) + C \leq C_0. \quad (2.4)$$

Let $K = C_0$ in (2.3), then $\|w_k\|^2 = 2C_0$, this is a contrary to (2.4).

Case b) Suppose $w \neq 0$, then by (2.2), we have

$$\int_{\Omega} |\Delta u_k|^2 dx - \lambda \int_{\Omega} \frac{|u_k|^2}{|x|^s} dx - \int_{\Omega} f(x, u_k) u_k dx = \langle I'(u_k), u_k \rangle = o(1), \forall \varphi \in H_0^2(\Omega).$$

Hence,

$$\|u_k\|^2 = \int_{\Omega} |\Delta u_k|^2 dx \geq \int_{\Omega} |\Delta u_k|^2 dx - \lambda \int_{\Omega} \frac{|u_k|^2}{|x|^s} dx = \int_{\Omega} f(x, u_k) u_k dx + o(1).$$

Let $\Omega_0 = \{x \in \Omega | w(x) = 0\}$, then $|\Omega \setminus \Omega_0| > 0$, and consequently from the inequality above, we have

$$1 \geq \int_{\Omega_0} f(x, u_k) \frac{u_k}{2K|u_k|^2} |w_k|^2 dx + \int_{\Omega \setminus \Omega_0} f(x, u_k) \frac{u_k}{2K|u_k|^2} |w_k|^2 dx + o(1). \quad (2.5)$$

Meanwhile, for any $x \in \Omega \setminus \Omega_0$, we have $|u_k|(x) \rightarrow +\infty$. Hence, by (f_3) , we get

$$f(x, u_k) \frac{u_k}{2K|u_k|^2} |w_k|^2 \rightarrow +\infty, k \rightarrow +\infty.$$

From $|\Omega \setminus \Omega_0| > 0$ and by Fatou Lemma, we have

$$\int_{\Omega_0} f(x, u_k) \frac{u_k}{2K|u_k|^2} |w_k|^2 dx \rightarrow +\infty, k \rightarrow +\infty. \quad (2.6)$$

On the other hand, by (f_3) , we know that there exists $M > -\infty$ such that $\frac{f(x,s)}{s} \geq M, \forall (x, s) \in \Omega \times \mathbb{R}$. And we also note that $\int_{\Omega_0} |w_k(x)|^2 dx \rightarrow 0$ as $k \rightarrow +\infty$. Furthermore, we can find $\Lambda > -\infty$ such that

$$\int_{\Omega_0} f(x, u_k) \frac{u_k}{2K|u_k|^2} |w_k|^2 dx \geq \frac{M}{2K} \int_{\Omega_0} |w_k|^2 dx > \Lambda > -\infty. \quad (2.7)$$

Therefore, from (2.5)-(2.7), we deduce a contradiction.

Thus, we obtain that functional I satisfies (C)-condition.

Lemma 2.5 *If $\alpha \in [1, 2^*(s))$, Then $\beta_k := \sup\{|u|_{\alpha} : u \in Z_k, \|u\| = 1\} \rightarrow 0 \quad (k \rightarrow +\infty)$.*

Proof It is clear that $0 < \beta_{k+1} \leq \beta_k$, so that $\beta_k \rightarrow \beta \geq 0$ as $k \rightarrow +\infty$. For every $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $|u_k|_{\alpha} > \beta_k/2$. By definition of Z_k , $u_k \rightarrow 0$ in $H_0^2(\Omega)$. The Sobolev imbedding theorem implies that $u_k \rightarrow 0$ in $L^p(\Omega)$. Thus we have proved that $\beta = 0$.

3 Proof of Theorem 1.1

The main results of this paper are stated as follows.

Proof of Theorem 1.1 Firstly, by (f_5) , we obtain that $I(-u) = I(u)$ for any $u \in H_0^2(\Omega)$, and then it is from Lemma 2.4 that I satisfies (C)-condition.

Next, it is from (f_2) that there exists $C > 0$ such that $|F(x, s)| \leq C(1 + |s|^q)$. Let $\beta_k := \sup\{|u|_q : u \in Z_k, \|u\| = 1\}$, $k = 1, 2, \dots$, then, by Lemma 2.5, $\beta_k \rightarrow 0$ as $k \rightarrow +\infty$. Set $r_k = \left(\frac{4C}{\sigma}\beta_k^q\right)^{\frac{1}{2-q}}$. If for $u \in Z_k$ such that $\|u\| = r_k$, then we have

$$I(u) \geq \frac{\sigma}{2}\|u\|^2 - C\beta_k^q\|u\|^q - C|\Omega| = \frac{\sigma}{2}r_k^2 - C\beta_k^q r_k^q - C|\Omega| = \frac{\sigma}{4}\left(\frac{4C}{\sigma}\right)^{\frac{2}{2-q}}\beta_k^{2q/(2-q)} - C|\Omega|,$$

Note that $\beta_k \rightarrow 0$ and $q > 2$, from the above inequality, we have

$$b_k = \inf_{u \in Z_k, \|u\|=r_k} I(u) \rightarrow +\infty, \text{ as } k \rightarrow +\infty.$$

Finally, we show that $a_k = \max_{u \in Y_k, \|u\|=\rho_k} I(u) \leq 0$. In fact, we define the norm $\|u\|_{\lambda,2}^2 = \int_{\Omega} (|\Delta u|^2 - \lambda \frac{|u|^2}{|x|^s}) dx$. Since on the finite dimensional space $\dim Y_k$ all norms are equivalent, we obtain that $\|u\|^2 \Leftrightarrow \|u\|_{\lambda,2}^2$. Hence, there exists $B_k, C_k > 0$ such that for any $u \in Y_k$,

$$\frac{1}{2} \int_{\Omega} (|\Delta u|^2 - \lambda \frac{|u|^2}{|x|^s}) dx = \frac{1}{2} \|u\|_{\lambda,2}^2 \leq C_k \|u\|_2^2 = C_k \int_{\Omega} |u|^2 dx, \quad (3.1)$$

$$\|u\|^2 \leq B_k \|u\|_2^2. \quad (3.2)$$

By (f_2) , there exists $R_k > 0$ such that $F(x, s) \geq 2C_k |s|^2$ as $|s| > R_k$. On the other hand, let $M_k = \max\{0, \inf_{x \in \Omega, |s| \leq R_k} F(x, s)\}$ if $|s| \leq R_k$, then for any $(x, s) \in \Omega \times \mathbb{R}$, we have

$$F(x, s) \geq 2C_k |s|^2 - M_k. \quad (3.3)$$

By (3.1)-(3.3), for any $u \in Y_k$, we have

$$I(u) = \frac{1}{2} \|u\|_{\lambda,2}^2 - \int_{\Omega} F(x, u) dx \leq -C_k \|u\|_2^2 + M_k |\Omega| \leq -\frac{C_k}{B_k} \|u\|^2 + M_k |\Omega|.$$

It is easy to see that there exists sufficiently large $\rho_k > 0$ such that $a_k = \max_{u \in Y_k, \|u\|=\rho_k} I(u) \leq 0$.

Therefore, by Lemma 2.1, functional I has an unbounded sequence of critical values.

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