Stability of a Paddy Ecosystem with Time Delay

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***Abstract*—A delay differential equation model of paddy ecosystem was put forward to reveal the interaction among rice, weeds and inorganic fertilizer on the system. The results show that, the system exists a rice and weed extinction equilibrium, and it also exists a rice extinction or weed extinction equilibrium. Their stable and unstable conditions are obtained. Moreover, Hopf bifurcations occur at the rice extinction or weed extinction equilibrium as the delay crosses some critical values. According to the conditions, some measures to increase rice yield were recommended.**

Keywords-paddy ecosystem; delay; equilibrium; stability; Hopf bifurcation

I. INTRODUCTION

As is well know, there are a lot of components in the paddy ecosystem, such as rice, weed, insect, microorganism, inorganic fertilizer, light intensity, moisture. The interaction of the components is a complex nonlinear relationship, and it also is a process of their survival, evolution and adaptation in the environment. At present, the research on paddy ecosystem is mainly concentrated in the field experiment and data analysis [1–3].

In order to describe quantitatively the change of material and energy in the paddy ecosystem, we need construct a dynamics model to reflect their conversion characteristics. Using this mathematical model, we can predict the yield of rice, we also can analyze the evolution law of soil fertility. But there are still not the matured mathematical models such as those in the forest and marine ecosystems [4-7].

The main purpose of this paper is to establish a differential equation model for the interaction among the main components of a paddy ecosystem, and to analyze the existence and stability of the system's equailibria.

II. THE MODELING OF A PADDY ECOSYSTEM

We only consider three main components of paddy ecosystem: rice, weed and inorganic fertilizer. The growth of rice and weeds are affected by soil fertility, light and other factors. There is natural death for the rice and weed. The inorganic fertilizer in soil partly comes from fertilization and partly comes from organic fertilizer such as decaying leaves of rice and weed, which can are transformed to inorganic fertilizer after some time by microbial. Natural loss also reduces the content of inorganic fertilizers in soil.

According to the interaction relationship, a dynamic model of composite farming paddy ecosystem is established as follows,

$$\begin{cases} \dot{r} = c_1 s_1 u(t) r(t) - d_1 r(t), \\ \dot{p} = c_2 s_2 u(t) p(t) - d_2 p(t), \\ \dot{u} = b + d_1 r(t-\tau) + d_2 p(t-\tau) \\ \qquad - s_1 u(t) r(t) - s_2 u(t) p(t) - d_3 u(t). \end{cases} \tag{1}$$

where $r(t)$ denotes the rice biomass per unit area at time t, $p(t)$ denotes the weed biomass per unit area at time t, and $u(t)$ denotes the inorganic fertilizer content per unit area at time t. In system (1), we assume that the uptake of inorganic fertilizer by rice and weed follows the mass action law.

On the right hand side of the first two equations in system (1), the c_1s_1ur and c_2s_2up represent rice and weed growth rate, respectively, and the d_1r and d_2p are the mortality of rice and weed. The rice and weed growth rates are affected by light intensity I and inorganic fertilizer u. The parameters s_i are a light effects (i=1,2).

On the right hand side of the third equation in system (1), the first term represents artificial fertilizer rate. The second and the third terms are inorganic fertilizers transformed from dead rice and weeds at t-τ time. The fourth and the fifth terms are the consumption of inorganic fertilizer by the growth of rice and weed. The last item indicates the loss of inorganic fertilizer in paddy field.

According to the actual requirements, the parameters in system (1) are non negative, and satisfy the following conditions $0<c_i\le 1$, $b\ge 0$ and $d_i>0$.

For the sake of convenience, we introduce the following notations,

$$\theta_1 = \frac{d_1}{c_1 s_1}, \theta_2 = \frac{d_2}{c_2 s_2}.$$

where θ_1 is called the relative mortality of rice, θ_2 is called the relative mortality of weed.

III. THE EXISTENCE AND STABILITY OF EQUILIBRIA

An equilibrium (r^*,p^*,u^*) of system (1) satisfies the following equations,

$$
\begin{aligned}
&F_1(r^*,p^*,u^*)=c_1s_1u^*r^*-d_1r^*=0,\\
&F_2(r^*,p^*,u^*)=c_2s_2u^*p^*-d_2p^*=0,\\
&F_3(r^*,p^*,u^*)=b+d_1r^*+d_2p^*-s_1u^*r^*-s_2u^*p^*-d_3u^*=0.
\end{aligned}
\tag{2}
$$

We make a coordinate transformation $x=r-r^*$, $y=p-p^*$, $z=u-u^*$, then system (1) can be converted to

$$
\begin{cases}
\dot{x}(t)=c_1s_1(u^*-\theta_1)x(t)+c_1s_1r^*z(t)+c_1s_1x(t)z(t),\\
\dot{y}(t)=c_2s_2(u^*-\theta_2)y(t)+c_2s_2p^*z(t)+c_2s_2y(t)z(t),\\
\dot{z}(t)=d_1x(t-\tau)-s_1u^*x(t)+d_2y(t-\tau)-s_2u^*y(t)\\
\qquad\quad -(s_1r^*+s_2p^*+d_3)z(t)-s_1x(t)z(t)-s_2y(t)z(t).
\end{cases}
\tag{3}
$$

So the characteristic equation of the linearized system of (1) is

$$
\begin{vmatrix}
\lambda-c_1s_1(u^*-\theta_1) & 0 & -c_1s_1r^*\\
0 & \lambda-c_2s_2(u^*-\theta_2) & -c_2s_2p^*\\
s_1u^*-d_1e^{-\lambda\tau} & s_2u^*-d_2e^{-\lambda\tau} & \lambda+s_1r^*+s_2p^*+d_3
\end{vmatrix}=0.
\tag{4}
$$

From $F_1(r^*,p^*,u^*)=0$, we have $r^*(c_1s_1u^*-d_1)=0$. Therefore, we obtain

$$r^*=0,\ \text{or}\ u^*=\theta_1.$$

From $F_2(r^*,p^*,u^*)=0$, we have $p^*(c_2s_2u^*-d_2)=0$. Therefore, we know

$$p^*=0,\ \text{or}\ u^*=\theta_2.$$

Thus we can calculate the equilibria of system (1) in three cases as follows.

Case (I), $r^*=0$ and $p^*=0$. At this case, we have the following conclusion of the existence and stability of equilibrium.

Theorem 1. There exists a paddy and weed extinction equilibrium of system (1), given by

$$(r_1^*,p_1^*,u_1^*)=\left(0,0,b/d_3\right).$$

If $b/d_3<\min\{\theta_1,\theta_2\}$, the equilibrium (r_1^*, p_1^*, u_1^*) is locally asymptotically stable. Otherwise, If $b/d_3>\min\{\theta_1,\theta_2\}$, the equilibrium (r_1^*, p_1^*, u_1^*) is unstable.

Proof. Substituting $r^*=0$ and $p^*=0$ into $F_3(r^*,p^*,u^*)=0$, we have $u^*=b/d_3$. Therefore, there exists a paddy and weed extinction equilibrium $(r_1^*, p_1^*, u_1^*)=(0,0, b/d_3)$ of system (1). At the equilibrium (r_1^*, p_1^*, u_1^*), the characteristic equation of the linearized system is as follows from (4)

$$
\begin{vmatrix}
\lambda-c_1s_1(u_1^*-\theta_1) & 0 & 0\\
0 & \lambda-c_2s_2(u_1^*-\theta_2) & 0\\
s_1u_1^*-d_1e^{-\lambda\tau} & s_2u_1^*-d_2e^{-\lambda\tau} & \lambda+d_3
\end{vmatrix}=0.
$$

It has three eigenvalues $\lambda_1=-d_3<0$, $\lambda_2=c_1s_1(b/d_3-\theta_1)$, and $\lambda_3=c_2s_2(b/d_3-\theta_2)$.

If $b/d_3<\min\{\theta_1,\theta_2\}$, the eigenvalues $\lambda_2<0$ and $\lambda_3<0$. Therefore, the equilibrium (r_1^*, p_1^*, u_1^*) of system (1) is locally asymptotically stable.

If $b/d_3>\min\{\theta_1,\theta_2\}$, at least one of the eigenvalues λ_2 and λ_3 is positive. Therefore, the equilibrium (r_1^*, p_1^*, u_1^*) is unstable under this condition.

Case (II), $p^*=0$ and $u^*=\theta_1$. From $F_3(r^*,p^*,u^*)=0$, we have

$$r^*=\frac{b-d_3\theta_1}{s_1\theta_1(1-c_1)}.$$

Therefore, we obtain the conclusion of the existences and stability of equilibrium at the case (II).

Theorem 2. If $b/d_3>\theta_1$, then system (1) has a equilibrium (r_2^*, p_2^*, u_2^*), where

$$r_2^*=\frac{b-d_3\theta_1}{s_1\theta_1(1-c_1)},\ p_2^*=0, u_2^*=\theta_1.$$

Furthermore, (I)if $\theta_1>\theta_2$, then the equilibrium (r_2^*, p_2^*, u_2^*) is unstable.

(II) If $\theta_1<\theta_2$ and

$$(s_1r_2^*+d_3)^2>2c_1s_1^2r_2^*\theta_1\left(1-\sqrt{1-c_1^2}\right),\tag{5}$$

then the equilibrium (r_2^*, p_2^*, u_2^*) is locally asymptotically stable for $\tau\geq 0$.

(III) If $\theta_1<\theta_2$ and

$$(s_1r_2^*+d_3)^2<2c_1s_1^2r_2^*\theta_1\left(1-\sqrt{1-c_1^2}\right),\tag{6}$$

then there exists a positive number τ_0, when $0\leq\tau<\tau_0$, the equilibrium (r_2^*, p_2^*, u_2^*) is locally asymptotically stable; when $\tau>\tau_0$, the equilibrium (r_2^*, p_2^*, u_2^*) is unstable, and a Hopf bifurcation emerges at $\tau=\tau_0$.

Proof. Obviously, r_2^* is positive if $b/d_3>\theta_1$. Therefore, there exists a weed extinct equilibrium (r_2^*, p_2^*, u_2^*).

We consider the stability of the equilibrium (r_2^*, p_2^*, u_2^*). From (4), the characteristic equation of the linearized system is as follows

$$(\lambda - c_2 s_2(\theta_1-\theta_2))(\lambda^2+(s_1 r_2^*+d_3)\lambda + c_1 s_1^2 r_2^* \theta_1(1-c_1 e^{-\lambda\tau}))=0.$$

It has one real eigenvalue $\lambda_1=c_2s_2(\theta_1-\theta_2)$. And its other eigenvalues are the roots of the following equation

$$\lambda^2+(s_1 r_2^*+d_3)\lambda + c_1 s_1^2 r_2^* \theta_1(1-c_1 e^{-\lambda\tau})=0. \quad (7)$$

(I) Obviously, if $\theta_1 > \theta_2$, the eigenvalue $\lambda_1 > 0$. It indicates that the equilibrium (r_2^*, p_2^*, u_2^*) is unstable.

(II) If $\theta_1 < \theta_2$, then the real eigenvalue $\lambda_1 < 0$. If the time delay $\tau = 0$, then the two roots of (7) are

$$\lambda_{2,3}=\frac{1}{2}\left(\begin{array}{l}-(s_1 r_2^*+d_3)\\ \pm\sqrt{(s_1 r_2^*+d_3)^2-4c_1 s_1^2 r_2^*\theta_1(1-c_1)}\end{array}\right).$$

Obviously, the real parts of $\lambda_{2,3}$ are less than zero. Therefore, when $\tau = 0$, the equilibrium (r_2^*, p_2^*, u_2^*) is locally asymptotically stable if $\theta_1 < \theta_2$.

Next we consider the case $\tau > 0$. Assume equation (7) has a imaginary root $\lambda=i\xi$ ($\xi>0$). Substituting it into (7) gives

$$-\xi^2+i(s_1 r_2^*+d_3)\xi + c_1 s_1^2 r_2^*\theta_1(1-c_1\cos\tau\xi + ic_1\sin\tau\xi)=0.$$

Separating its real and imaginary parts yields

$$-\xi^2+c_1 s_1^2 r_2^*\theta_1(1-c_1\cos\tau\xi)=0,$$

and

$$(s_1 r_2^*+d_3)\xi + c_1^2 s_1^2 r_2^*\theta_1\sin\tau\xi=0.$$

So we have

$$\cos\tau\xi=\frac{c_1 s_1^2 r_2^*\theta_1-\xi^2}{c_1^2 s_1^2 r_2^*\theta_1},\ \sin\tau\xi=-\frac{(s_1 r_2^*+d_3)\xi}{c_1^2 s_1^2 r_2^*\theta_1}. \quad (8)$$

Based on $\sin^2\tau\xi + \cos^2\tau\xi = 1$, one has

$$(c_1 s_1^2 r_2^*\theta_1-\xi^2)^2+(s_1 r_2^*+d_3)^2\xi^2=c_1^4 s_1^4 r_2^{*2}\theta_1^2.$$

That is

$$\xi^4+\left((s_1 r_2^*+d_3)^2-2c_1 s_1^2 r_2^*\theta_1\right)\xi^2 + c_1^2 s_1^4 r_2^{*2}\theta_1^2(1-c_1^2)=0. \quad (9)$$

From (5), we have

$$2c_1 s_1^2 r_2^*\theta_1-(s_1 r_2^*+d_3)^2<2c_1 s_1^2 r_2^*\theta_1\sqrt{1-c_1^2}. \quad (10)$$

Obviously, if $(s_1r_2^*+d_3)^2 - 2c_1s_1^2r_2^*\theta_1 \geq 0$, then there is not any real number ξ such that (9) holds. Otherwise, if $(s_1r_2^*+d_3)^2 - 2c_1s_1^2r_2^*\theta_1 < 0$, then taking square on the two sides of (10) yields

$$\left(2c_1 s_1^2 r_2^*\theta_1-(s_1 r_2^*+d_3)^2\right)^2-4c_1^2 s_1^4 r_2^{*2}\theta_1^2\left(1-c_1^2\right)<0.$$

So, there also is not any real number ξ to make (9) hold. Therefore, the real parts of any roots of (7) must be negative for any $\tau > 0$. It shows that the equilibrium (r_2^*, p_2^*, u_2^*) is locally asymptotically stable for any $\tau \geq 0$.

(III) If condition (6) holds, we have

$$2c_1 s_1^2 r_2^*\theta_1-(s_1 r_2^*+d_3)^2>2c_1 s_1^2 r_2^*\theta_1\sqrt{1-c_1^2}.$$

Taking square on the two sides yields

$$\Delta=\left(2c_1 s_1^2 r_2^*\theta_1-(s_1 r_2^*+d_3)^2\right)^2-4c_1^2 s_1^4 r_2^{*2}\theta_1^2\left(1-c_1^2\right)>0.$$

Therefore, there exist two positive real number ξ_- and ξ_+ such that (9) holds, where one might as well assume $\xi_- \leq \xi_+$. Thus,

$$\xi_\pm^2=\frac{1}{2}\left(2c_1 s_1^2 r_2^*\theta_1-(s_1 r_2^*+d_3)^2\pm\sqrt{\Delta}\right). \quad (11)$$

From (8), it is easy to know that $\cos\tau_-\xi_- > 0$. Noticed that $\sin\tau_-\xi_- < 0$, so

$$\tau_-=\frac{2\pi}{\xi_-}-\frac{1}{\xi_-}\arcsin\frac{(s_1 r_2^*+d_3)\xi_-}{c_1^2 s_1^2 r_2^*\theta_1}.$$

If $\cos\tau_+\xi_+ > 0$, and noticed that $\sin\tau_+\xi_+ < 0$, then we have

$$\tau_+=\frac{2\pi}{\xi_+}-\frac{1}{\xi_+}\arcsin\frac{(s_1 r_2^*+d_3)\xi_+}{c_1^2 s_1^2 r_2^*\theta_1}.$$

Therefore,

$$\tau_0=\min\{\tau_-,\tau_+\}=\frac{2\pi}{\xi_+}-\frac{1}{\xi_+}\arcsin\frac{(s_1r_2^*+d_3)\xi_+}{c_1^2s_1^2r_2^*\theta_1}. \quad (12)$$

If $\cos\tau_+\xi_+<0$, and $\sin\tau_+\xi_+<0$, then we have

$$\tau_+=\frac{\pi}{\xi_+}+\frac{1}{\xi_+}\arcsin\frac{(s_1r_2^*+d_3)\xi_+}{c_1^2s_1^2r_2^*\theta_1}.$$

Therefore,

$$\tau_0=\min\{\tau_-,\tau_+\}=\frac{\pi}{\xi_+}+\frac{1}{\xi_+}\arcsin\frac{(s_1r_2^*+d_3)\xi_+}{c_1^2s_1^2r_2^*\theta_1}. \quad (13)$$

Next, we verify the transversal condition. Taking the derivative of λ with respect to τ in (7), we have

$$\frac{d\lambda}{d\tau}=-\frac{\lambda c_1^2s_1^2r_2^*\theta_1e^{-\lambda\tau}}{2\lambda+s_1r_2^*+d_3+c_1^2s_1^2r_2^*\theta_1\tau e^{-\lambda\tau}}.$$

From (11), we obtain

$$\mathrm{Re}\left\{\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\xi_+}\right\}=\frac{2\xi_+^2+\left(s_1r_2^*+d_3\right)^2-2c_1s_1^2r_2^*\theta_1}{\left(c_1^2s_1^2r_2^*\theta_1\right)^2}=\frac{\sqrt{\Delta}}{\left(c_1^2s_1^2r_2^*\theta_1\right)^2}>0.$$

Thus, the transversal condition is satisfied, hence a Hopf bifurcation occurs at $\tau=\tau_0$.

Case (III), p=0 and $u=\theta_1$. Similar to case (II), we have the following conclusion of the existence and stability of equilibrium.

Theorem 3. If $b/d_3>\theta_2$, then system (1) has a equilibrium (r_3^*, p_3^*, u_3^*), where

$$r_3^*=0,\ p_3^*=\frac{b-d_3\theta_2}{s_2\theta_2(1-c_2)},\ u_3^*=\theta_2.$$

Furthermore, (I) if $\theta_1<\theta_2$, then the equilibrium (r_3^*, p_3^*, u_3^*) is unstable.

(II) If $\theta_1>\theta_2$ and

$$(s_2p_3^*+d_3)^2>2c_2s_2^2p_3^*\theta_2\left(1-\sqrt{1-c_2^2}\right), \quad (14)$$

then the equilibrium (r_3^*, p_3^*, u_3^*) is locally asymptotically stable for $\tau\ge 0$.

(III) If $\theta_1>\theta_2$ and

$$(s_2p_3^*+d_3)^2<2c_2s_2^2p_3^*\theta_2\left(1-\sqrt{1-c_2^2}\right), \quad (15)$$

then there exist a positive number τ_0, when $0\le\tau<\tau_0$, the equilibrium (r_3^*, p_3^*, u_3^*) is locally asymptotically stable; when $\tau>\tau_0$, the equilibrium (r_3^*, p_3^*, u_3^*) is unstable, and a Hopf bifurcation emerges at $\tau=\tau_0$.

IV. EXAMPLES

According to those discussion in Section III, we give two examples to illustrate the correctness of our results.

Example 1. In system (1), let c_1=0.8, c_2=0.3, s_1=6, s_2=2, b=5, d_1=0.2, d_2=0.7, d_3=0.2 and τ = 5. Then system (1) has three equilibria as follows, the paddy and weed extinct equilibrium (0, 0, 25), the paddy extinct equilibrium (0, 2.9184, 1.1667), and the weed extinct equilibrium ((99.8333, 0, 0.0417).

By computing, we have b/d_3 = 25, $\theta_1\approx 0.0417$ and $\theta_2\approx 1.1667$. So the inequality $b/d_3>\theta_2>\theta_1$ holds. We can verify inequalitie (5) hold. From Theorem 1-3, the equilibria (0, 0, 25) and (0, 2.9184, 1.1667) are unstable, and the weed extinction equilibrium (99.8333, 0, 0.0417) is asymptotically stable for any $\tau\ge 0$.

Example 2. In system (1), let c_1=0.5, c_2=0.3, s_1=0.6, s_2=0.01, b=0.015, d_1=0.2, d_2=0.99, d_3=0.01 and τ = 5. By computing, we have b/d_3 = 1.5, $\theta_1\approx 0.6667$ and $\theta_2\approx 330$. So the inequality $\theta_2>b/d_3>\theta_1$ holds. Then system (1) has two equilibria as follows, the paddy and weed extinct equilibrium (0, 0, 1.5), and the weed extinct equilibrium (0.0417, 0, 0.6667).

It is not difficult to verify inequality (6) holds. From (13), we obtain $\tau_0\approx 65.4899$. Therefore, by Theorem 2, the equilibria (0.0417, 0, 0.6667) is asymptotically stable when $0\le\tau<\tau_0$ (see Fig. 1(A)); when $\tau>\tau_0$, the equilibrium (0.0417, 0, 0.6667) is unstable, and a Hopf bifurcation emerges at $\tau=\tau_0$. (see Fig. 1(B)).

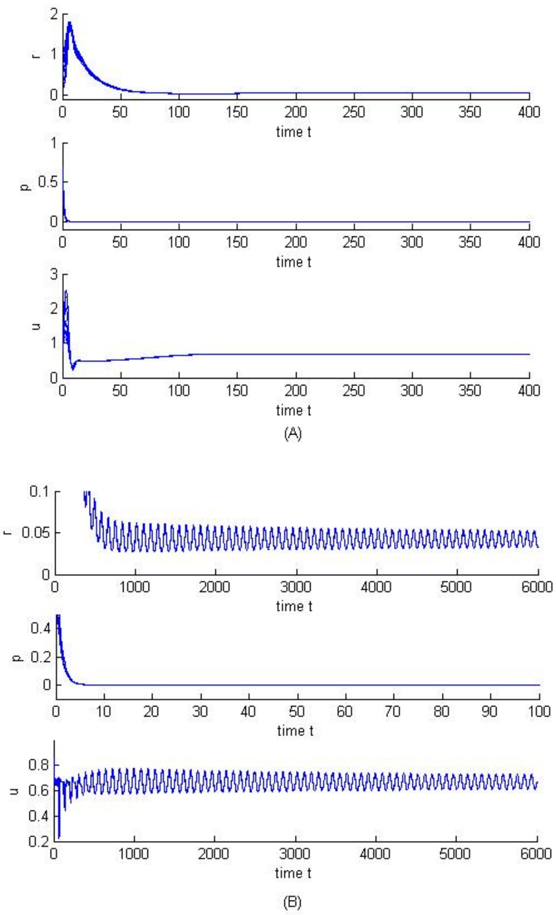

FIGURE I. TIME RESPONSE CURVES OF EXAMPLE 2, C1=0.5, C2=0.3, S1=0.6, S2=0.01, B=0.015, D1=0.2, D2=0.99 AND D3=0.01. (A) T=5, (B) T=65.5

V. CONCLUSIONS

We have proposed a differential equation model that reflects the interaction among rice, weed and inorganic fertilizer in the paddy ecosystem. In this system, a rice and weed extinct equilibrium (0, 0, u_1^*) always exists, and the equilibrium is stable under condition $b/d_3 < \min\{\theta_1, \theta_2\}$. Otherwise, it is unstable. At this time, if $\theta_1 > \theta_2$, then when $b/d_3 > \theta_2$, system (1) has a rice extinction equilibrium (0, p_2^*, u_2^*), which is asymptotically stable; when $b/d_3 > \theta_1$, system (1) still has a weed extinction equilibrium (r_3^*, 0, u_3^*), which is unstable. If $\theta_1 < \theta_2$, then when $b/d_3 > \theta_1$, system (1) has a weed extinction equilibrium (r_3^*, 0, u_3^*), which is asymptotically stable, when $b/d_3 > \theta_2$, system (1) still has a rice extinction equilibrium (0, p_2^*, u_2^*), which is unstable. Our results show that the existence and stability of equilibrium points are related to the relative mortality of rice and weed, θ_1 and θ_2, and to the ratio of fertilizer supply and loss b/d_3. The presence of time delay τ maybe drive the system to instability.

In the paddy field management, we should take some measures to make the weed extinction equilibrium (r_3^*, 0, u_3^*), exist and be stable. Obviously, at this time the equilibria (0, 0, u_1^*) and (0, p_2^*, u_2^*) are not stable or not exist. According to Theorem 3, these measures include: reducing the loss rate d_3 of inorganic fertilizer, increasing fertilization rate b, selecting rice varieties with low mortality d_1, increasing mortality of weed d2, reducing t the utilization rate of light energy of weed. We also can reduce the rice mortality rate d_1, increase the utilization rate of light energy of rice c_1. From the expression of r_3^*, these measures also help to increase rice yield.

ACKNOWLEDGMENT

This work was supported in part by Hunan province science and technology project (grants 2015JC3101) and Hunan province graduate student innovation training project (grants CX2015B265).

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