The Value of European Option

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Abstract—**In this paper we study the patterns of behavior of the stock price and the results of Merton on European option pricing spread by stochastic analysis method. Assume that the stock price jump process is special class of compound Poisson process and the volatility without jump is the function of time. We derived the European option with continuous dividends pricing formula under the assumption of risk neutral and stock price jump process for of the compound Poisson process, to promote the results of Merton.**

Keywords-option pricing; compound Poisson process; jump-diffusion process

I. INTRODUCTION

Modern option pricing theory revolution began in 1973, F.Black M.Scholes in assumptions effectively market and stock prices follow Ito process, derive the famous Black-Scholes option pricing model [1]. In reality, however, some significant information arrives to lead to stock price discontinuous changes that jump, Merton jump diffusion model [2] established in 1976, which the diffusion process represents the continuous fluctuations of stock prices jump process said not continuous fluctuations of stock prices, and he assumed that the jump process with a Poisson process. In recent years, there are still many people in this area for further research [3-5], this article assumes that the process of jumping to a special class of compound Poisson process , we study European options with continuous dividends, and to disseminate the results of the literature [2].

II. ASSUMPTIONS AND MODEL

Definition1. Suppose that $Y_1, Y_2, \cdots$ is a sequence of i.i.d. random variables, $\{N_t, t \geq 0\}$ is a Poisson process with intensity parameter λ , and independent of $\{Y_n, n \geq 1\}$, let $X_t = \sum_{i=1}^{N_t} Y_i$, then $\{X_t, t \geq 0\}$ is call the compound Poisson process.

In this paper we consider that a kind of special compound Poisson process who $Y_i - 1$ are uniform distribution sequence whose possible values are $\{0, 1, 2, \cdots, m\}$.

Proposition 1. Suppose that $\{X_t, t \geq 0\}$ is a kind of special compound Poisson process described above, then

$$P(X_t = 0) = \mathrm{e}^{-\lambda t},$$

$$P(X_t = n) = \sum_{j=0}^{n} \frac{(\lambda t)^j \mathrm{e}^{-\lambda t}}{j!} \left[G^{(j)}(n) - G^{(j)}(n-1) \right],$$

$$n = 1, 2, \cdots \tag{1}$$

Especially $m = 0$, $\{X_t\}$ is Poisson process. Where $G(y)$ is the distribution function of Y_1, $G^{(j)}(y)$ is j fold convolution of $G(y)$ with itself, and conventions $G^{(0)}(n) = 1$.

Proof.

$$P(X_t \leq n) = \sum_{j=0}^{\infty} P\{\sum_{i=1}^{N_t} Y_i \leq n \big| N_t = j\} P(N_t = j)$$

$$= \sum_{j=0}^{\infty} P\{\sum_{i=1}^{j} Y_i \leq n\} \frac{(\lambda t)^j \mathrm{e}^{-\lambda t}}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{(\lambda t)^j \mathrm{e}^{-\lambda t}}{j!} G^{(j)}(n)$$

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So

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$$P(X_t = n) = P(X_t \leq n) - P(X_t \leq n-1)$$

$$= \sum_{j=0}^{n} \frac{(\lambda t)^j \mathrm{e}^{-\lambda t}}{j!} \left[G^{(j)}(n) - G^{(j)}(n-1) \right]$$

Let $(\Omega, F, P, (\mathrm{F}_t)_{0 \leq t \leq T})$ be a probability space. We consider a frictionless financial market consisting of a riskless bond M_t and a risky asset S_t , that are traded up to a finite time horizon T ,

We suppose that M_t and S_t satisfy the differential equation

$$\mathrm{d}M_t = M_t r\mathrm{d}t, \quad M_0 = 1, \tag{2}$$

$$\mathrm{d}S_t = S_t((r-\rho(t))\mathrm{d}t - \mu \mathrm{d}\sum_{n=0}^{\infty} nP_n(t) + \sigma(t)\mathrm{d}W_t + U\mathrm{d}X_t) \tag{3}$$

where risk-free interest rate r, volatility $\sigma(t)$ and continuous dividend $\rho(t)$, $\{W(t), 0 \le t \le T\}$ be a standard Wiener process, $\mu = E(U)$,We assume that the filtration $(\mathrm{F}_t, 0 \le t \le T)$ is generated by the S_t and martingale X_t.

III. Option Pricing Formula

Proposition 2 Assume S_t satisfy the stochastic differential equation (3) stock price behavior, maturity date T, exercise price K, then at time t the price $C(t, S_t)$ of European call option satisfy

$$C(t,S_t) = \sum_{k=0}^{\infty} P_k(T-t)E[S_t\Phi(d_1)\bullet$$

$$\exp\{-\mu\sum_{n=0}^{\infty} nP_n(T-t)\}\prod_{i=1}^{k}(1+U_i) - K\mathrm{e}^{-r(T-t)}\Phi(d_2)]$$

where

$$d_1 = \{\ln\frac{S_t\prod_{i=1}^{k}(1+U_i)}{K} + r(T-t)$$

$$-\mu\sum_{n=1}^{\infty} nP_n(T-t) + \int_t^T \frac{1}{2}\sigma^2(s) - \rho(s)\mathrm{d}s\}$$
$$\div\sqrt{\int_t^T \sigma^2(s)\mathrm{d}s},$$

$$d_2 = d_1 - \sqrt{\int_t^T \sigma^2(s)\mathrm{d}s}.$$

Proof. Since $\tilde{S}_t = \mathrm{e}^{-rt}S_t$ is the martingale under risk-neutral martingale measure, then

$$C(t,S_t) = E[\mathrm{e}^{-r(T-t)}(S_t-K)^+ |\mathbf{F}_t]$$

$$= E\left[\mathrm{e}^{-r(T-t)}S_T\chi_{\{S_T\ge K\}}|\mathbf{F}_t\right] - KE\left[\mathrm{e}^{-r(T-t)}\chi_{\{S_T\ge K\}}|\mathbf{F}_t\right]. \tag{4}$$

Let $\frac{\mathrm{d}P^*}{\mathrm{d}P}\Big|\mathbf{F}_t = \mathrm{e}^{-r(T-t)}\frac{S_T}{S_t}$, as well as "Bayes's rule", we may

$$E[\mathrm{e}^{-r(T-t)}S_T\chi_{\{S_T\ge K\}}|\mathbf{F}_t]$$
$$= E^*[S_t\chi_{\{S_T\ge K\}}|\mathbf{F}_t]E[\mathrm{e}^{-r(T-t)}\frac{S_T}{S_t}|\mathbf{F}_t] = S_tP^*(S_T\ge K). \tag{5}$$

Let $Z_t = \frac{M_t}{S_t}$, applying Ito's lemma yield, we have

$$\mathrm{d}Z_t = Z_t((\sigma^2(t)+\rho(t))\mathrm{d}t$$

$$+\mu\mathrm{d}\sum_{n=1}^{\infty} nP_n(t) - \sigma(t)\mathrm{d}W_t - U\mathrm{d}X_t) \tag{6}$$

According to Girsanov theorem, $B_t = W_t - \int_0^t \sigma(s)\mathrm{d}s$ is a standard Wiener process under the measure P^*, then(6)equivalently

$$\mathrm{d}Z_t = Z_t(\rho(t)\mathrm{d}t + \mu\mathrm{d}\sum_{n=1}^{\infty} nP_n(t) - \sigma(t)\mathrm{d}B_t - U\mathrm{d}X_t) \tag{7}$$

Equation (7) exists unique solution

$$Z_T = Z_t\prod_{i=1}^{X_{T-t}}\frac{1}{(1+U_i)}\exp\{\int_t^T \rho(s) - \frac{1}{2}\sigma^2(s)\mathrm{d}s$$

$$+\mu\sum_{n=1}^{\infty} nP_n(T-t) - \int_t^T \sigma(s)\mathrm{d}B_s\}$$

We have

$$P^*(S_T\ge K) = P^*(\ln Z_T \le \ln\frac{M_T}{K})$$

$$= P^*(\ln\{Z_t\prod_{i=1}^{X_{T-t}}\frac{1}{(1+U_i)}\exp\{\int_t^T \rho(s) - \frac{1}{2}\sigma^2(s)\mathrm{d}s +$$

$$\mu\sum_{n=1}^{\infty} nP_n(T-t) - \int_t^T \sigma(s)\mathrm{d}B_s\}\} \le \ln\frac{M_T}{K})$$

$$= P^*(-\int_t^T \sigma(s)\mathrm{d}B_s \le \ln \frac{S_t \prod_{i=1}^{X_{T-t}} (1+U_i)}{K} + r(T-t) -$$

$$\mu \sum_{n=1}^{\infty} nP_n(T-t) + \int_t^T \frac{1}{2}\sigma^2(s) - \rho(s)\mathrm{d}s)$$

$$= \sum_{k=0}^{\infty} P_k(T-t) E[\exp\{-\mu \sum_{n=1}^{\infty} nP_n(T-t)\} \prod_{i=1}^{k} (1+U_i)\Phi(d_1)]. \tag{8}$$

By virtue of (5)and(8)

$$E[\mathrm{e}^{-r(T-t)} S_T \chi_{\{S_T \ge K\}} | \mathbf{F}_t] = S_t \sum_{k=0}^{\infty} P_k(T-t)$$

$$E[\exp\{-\mu \sum_{n=1}^{\infty} nP_n(T-t)\} \prod_{i=1}^{k} (1+U_i)\Phi(d_1)] \tag{9}$$

Use the same method can be

$$KE\left[\mathrm{e}^{-r(T-t)} \chi_{\{S_T \ge K\}} | \mathbf{F}_t \right]$$

$$= K\mathrm{e}^{-r(T-t)} \sum_{k=0}^{\infty} P_k(T-t) E[\Phi(d_2)]. \tag{10}$$

Together with (4), (8) and (10), we have

$$C(t, S_t) = \sum_{k=0}^{\infty} P_k(T-t) E[S_t \Phi(d_1) \bullet$$

$$\exp\{-\mu \sum_{n=0}^{\infty} nP_n(T-t)\} \prod_{i=1}^{k} (1+U_i) - K\mathrm{e}^{-r(T-t)} \Phi(d_2)]$$

Proposition 3 Assume S_t satisfy the stochastic differential equation (3) stock price behavior, maturity date T, exercise price K. If U have lognormal distribution with mean parameter μ and variance σ^2 under the martingale measure P ,then at time t the price $C(t, S_t)$ of European call option satisfy

$$C(t, S_t) = \sum_{k=0}^{\infty} P_k(T-t) [S_t \mu^k \exp\{-\mu \sum_{n=0}^{\infty} nP_n(T-t)\}$$

$$\Phi(a_1) - K\mathrm{e}^{-r(T-t)} \Phi(a_2)]$$

where

$$a_1 = [\ln \frac{S_t}{K} + r(T-t) - \mu \sum_{n=1}^{\infty} nP_n(T-t)$$

$$+ \int_t^T \frac{1}{2}\sigma^2(s) - \rho(s)\mathrm{d}s + k\mu]$$

$$\div \sqrt{\int_t^T \sigma^2(s)\mathrm{d}s + k\sigma^2},$$

$$a_2 = a_1 - \sqrt{\int_t^T \sigma^2(s)\mathrm{d}s + k\sigma^2}.$$

Proof We have $-\int_t^T \sigma(s)\mathrm{d}B_s - \sum_{i=1}^{k} (1+U_i)$ would be a normal random variable under the martingale measure with mean and variance given by

$$E[-\int_t^T \sigma(s)\mathrm{d}B_s - \sum_{i=1}^{k} (1+U_i)] = -k\mu,$$

$$D[-\int_t^T \sigma(s)\mathrm{d}B_s - \sum_{i=1}^{k} (1+U_i)] = \int_t^T \sigma^2(s)\mathrm{d}s + k\sigma^2,$$

then

$$E[\prod_{i=1}^{k} (1+U_i)\Phi(d_1)] = \mu^k E[\Phi(d_1)]$$

$$= \mu^k P(-\int_t^T \sigma(s)\mathrm{d}B_s \le \ln \frac{S_t \prod_{i=1}^{k} (1+U_i)}{K} + r(T-t) -$$

$$\mu \sum_{n=1}^{\infty} nP_n(T-t) + \int_t^T \frac{1}{2}\sigma^2(s) - \rho(s)\mathrm{d}s)$$

$$= \mu^k P(-\int_t^T \sigma(s)\mathrm{d}B_s - \sum_{i=1}^{k} (1+U_i) \le \ln \frac{S_t}{K} + r(T-t) -$$

$$\mu \sum_{n=1}^{\infty} nP_n(T-t) + \int_t^T \frac{1}{2}\sigma^2(s) - \rho(s)\mathrm{d}s)$$

$$= \mu^k P(-\int_t^T \sigma(s)\mathrm{d}B_s - \sum_{i=1}^{k} (1+U_i) \le a)$$

$$= \mu^k \Phi(\frac{a + k\mu}{\sqrt{\int_t^T \sigma^2(s)\mathrm{d}s + k\sigma^2}})$$

$$= \mu^k \Phi(a_1)$$

where

$$a = \ln\frac{S_t}{K} + r(T-t) - \mu\sum_{n=1}^{\infty} nP_n(T-t) + \int_t^T \frac{1}{2}\sigma^2(s) - \rho(s)\mathrm{d}s$$

Similarly, we have

$$\Phi(d_2) = \Phi(a_1 - \sqrt{\int_t^T \sigma^2(s)\mathrm{d}s + k\sigma^2})$$

Using **Proposition 2**, we have

$$C(t,S_t) = \sum_{k=0}^{\infty} P_k(T-t)[S_t\mu^k \exp\{-\mu\sum_{n=0}^{\infty} nP_n(T-t)\}$$

$$\Phi(a_1) - K\mathrm{e}^{-r(T-t)}\Phi(a_2)]$$

Proposition 4 Assume S_t satisfy the stochastic differential equation (3) stock price behavior, maturity date T, exercise price K. Then the put-call parity relation may be rewritten as

$$C(t,S_T) - P(t,S_T) = S_t - Ke^{-r(T-t)}.$$

Proof

$$C(t,S_T) - P(t,S_T)$$

$$= e^{-r(T-t)}E[(S_T-K)^+ - (K-S_T)^+ \,|\, F_t]$$

$$= e^{-r(T-t)}E[S_T - K \,|\, F_t]$$

$$= E[e^{-r(T-t)}S_T \,|\, F_t] - Ke^{-r(T-t)}$$

$$= S_t - Ke^{-r(T-t)}$$

We can using put-call parity to find the price of a European put option on a stock with the same parameters as earlier.

Proposition 5 Assume S_t satisfy the stochastic differential equation (3) stock price behavior, maturity date T, exercise price K, then at time t the price $\mathrm{P}(t,S_t)$ of European put option satisfy

$$\mathrm{P}(t,S_t) = \sum_{k=0}^{\infty} P_k(T-t)E[K\mathrm{e}^{-r(T-t)}\Phi(-d_2) -$$

$$S_t\Phi(-d_1)\exp\{-V\sum_{n=0}^{\infty} nP_n(T-t)\}\prod_{i=1}^{k}(1+U_i)]$$

Especially $m=0$, $\{X_t\}$ is Poisson process, when $\sigma(t)$ is a constant and $\rho(t)=0$, **Proposition** is Merton the famous jump diffusion model results.

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