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The Pricing of European Exchange Option

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***Abstract*—This paper discusses the problem of pricing on some multi-asset option European exchange option in jump-diffusion model by martingale method. By changing basic assumption of William Margrabe exchange option pricing model to the assumption that jump process is count process that more general than Poisson process, it is established that the behavior model of the stock pricing process is jump-diffusion process. The formula of European exchange option whose stock price with jump process is a count process that more general than Poisson process is deduced under the risk-neutral hypothesis, and it is extended that William Margrabe exchange option pricing model.**

Keywords-European exchange options; jump-diffusion; count process

I. INTRODUCTION

This Throughout the nineties, we have seen the synergistic union of mathematics, finance, the computer, and the global economy. Currency markets trade tow trillion dollars per day, and sophisticated financial derivatives such as options, swaps, and quantors are commonplace. Since the appearance of the Black-Scholes formula in 1973[1], the financial community has embraced an abundant and ever-expanding array of mathematical tolls and models. Continuous-time mathematics has become one of the essential tools of modern finance. The elegant mathematics of stochastic calculus simplifies the solution of a wide range of important problems in finance. William Margrabe(1978) [2] study an equation for the value of the option to exchange one risky asset for another. His theory grows out of the brilliant Black-Scholes(1973) solution to the longstanding call option pricing problem---which assumes that the price of a riskless discount bond grew exponentially at the riskless interest rate---and Merton's(1973) [3] extension---in which the discount bond's value is stochastic until maturity. But we show that real data cannot always be fit by a geometric Brownian motion model, and that more general models may need to be considered. The appearance of important information will cause the stock price to a kind of not continual jumps[4-6].A mass of finance practice has indicated that there is a serious warp between the hypothesis of Black-Scholes model about the underlying asset price and the realistic markets. Therefore, many scholars put forward many new kinds of option pricing models by relaxing some assuming conditions of Black-Scholes model. Option pricing theory with jump-diffusion is one of them. In this paper, I develop an equation for the value of the option to exchange one risky asset for another. Establish the option-pricing model when exercise price is random variable. The option-pricing model is options to exchange one asset to another. Pricing formula of European option is also given.

II. ASSUMPTIONS AND MODELS

Let $(\Omega, F, P, (F_t)_{0\le t\le T})$ be a probability space and $\{W_t^2, W_t^3, 0\le t\le T\}$ be a two-dimensional standard Wiener process given on a probability space (Ω, F, P). The market is built with a bond B_t and two risky assets S_t^1, S_t^2. We suppose that B_t is the solution of the equation

$$\frac{dB_t}{B_t} = rdt \; ; \; B_0 = 1 , \tag{1}$$

and S_t^1, S_t^2 satisfies the stochastic differential equation

$$\frac{dS_t^1}{S_{t^-}^1} = rdt + \sigma_1 dW_t^1 + (V-1)d\tilde{N}_t \tag{2}$$

$$\frac{dS_t^2}{S_{t^-}^2} = rdt + \sigma_2 dW_t^2 + (U-1)d\tilde{N}_t \tag{3}$$

where risk-free interest rate r and volatility σ_1, σ_2, are supposed to be constant. $\{W_t^1, 0\le t\le T\}$ ($\{W_t^1 = \rho W_t^2 + \sqrt{1-\rho^2}W_t^3, 0\le t\le T\}$), $\{W_t^2, 0\le t\le T\}$ are standard Wiener process on a suitable probability space (Ω, F, P) .The correlation between the $\{W_t^1, 0\le t\le T\}$ and $\{W_t^2, 0\le t\le T\}$ is ρ, $\{\tilde{N}_t, 0\le t\le T\}$ is the compensated martingale of nonexplosive counting process[7] $\{N_t, 0\le t\le T\}$ with intensity parameter $\lambda(t)$ and $V_0, V_1, V_2, \cdots V_{N_t}$ $(V_i > 0, V_0 = 0)$ (V_i is independent of V_j, for $i\ne j$), $U_0, U_1, U_2, \cdots U_{N_t}$ $(U_i > 0, U_0 = 0)$ (U_i is independent of U_j ,for $i\ne j$)are the random variable percentage change in the money supply if the count process occurs. The random variables $\{W_t^2, 0\le t\le T\}$, $\{W_t^3, 0\le t\le T\}$, $\{N_t, 0\le t\le T\}$ and $\{V_i, 1\le i\le N_t\}$, $\{U_i, 1\le i\le N_t\}$ are assumed to be mutually independent.

We assume that the filtration $(F_t, 0\le t\le T)$ is generated by the martingale $\{\tilde{N}_t, 0\le t\le T\}$ and $\{W_t^1, 0\le t\le T\}$, $\{W_t^2, 0\le t\le T\}$.

Assumption 1 P is risk-neutral martingale measure[8].

Assumption 2 $\sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2 \ne 0$.

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Lemma 1 Let $\frac{dP^*}{dP}=\prod_{i=1}^{N_T}V_i\exp\{-\frac{1}{2}\sigma T+\sigma W_T+(1-E(V))\int_0^T\lambda(s)ds\}$, then, under the martingale measure P^*, process $\{N_t,0\le t\le T\}$ is nonexplosive counting process with intensity parameter $E(V)\lambda(t)$, and

$$P_n^*(T)=P_n(T)(E(V))^n\exp\{(1-E(V))\int_0^T\lambda(s)ds\}$$

$$E_*[f(V_1,V_2,\cdots,V_n)]=\frac{1}{(E(V))^n}E[f(V_1,V_2,\cdots,V_n)\cdot\prod_{i=1}^{n}V_i]$$

If the jumps $V_i,i\ge 1$ have a lognormal distribution with mean parameter μ_0 and variance $\sigma_0{}^2$ under the martingale measure P ,then $\ln V_i,i\ge 1$ have a lognormal distribution with mean parameter $\mu_0+\sigma_0{}^2$ and variance $\sigma_0{}^2$ under the martingale measure P^*.

Proof

$$\begin{aligned}P_n^*(T)&=E_*[I_{\{N_T=n\}}]\ =E[\frac{dP^*}{dP}I_{\{N_T=n\}}]\\&=E[\exp\{\sigma W_T-\frac{1}{2}\sigma^2T\}\cdot\exp\{(1-E(V))\int_0^T\lambda(s)ds\}\prod_{i=1}^{n}V_iI_{\{N_T=n\}}]\\&=P_n(T)(E(V))^n\exp\{(1-E(V))\int_0^T\lambda(s)ds\}\end{aligned}$$

$$\begin{aligned}E_*[f(V_1,V_2,\cdots,V_n)]&=E[f(V_1,V_2,\cdots,V_n)\frac{dP^*}{dP}]\\&=E[f(V_1,V_2,\cdots,V_n)\exp\{(1-E(V))\int_0^T\lambda(s)ds\}\prod_{i=1}^{N_T}V_i]\\&=E[\exp\{(1-E(V))\int_0^T\lambda(s)ds\}\prod_{i=n+1}^{N_T}V_i]E[f(V_1,V_2,\cdots,V_n)\prod_{i=1}^{n}V_i]\\&=\frac{1}{(E(V))^n}E[f(V_1,V_2,\cdots,V_n)\cdot\prod_{i=1}^{n}V_i]\end{aligned}$$

If the jumps $V_i,i\ge 1$ have a lognormal distribution with mean parameter μ_0 and variance $\sigma_0{}^2$ under the martingale measure P ,then

$$\begin{aligned}P^*(\ln V_i<x)&=E^*(I_{\{\ln V_i<x\}})\\&=E[\exp\{(1-E(V)\int_0^T\lambda(s)ds\}\prod_{i=1}^{N_T}V_iI_{\{\ln V_i<x\}}]\\&=\frac{E[V_iI_{\{\ln V_i<x\}}]}{E(V)}\\&=\exp\{-\mu_0-\frac{1}{2}\sigma_0{}^2\}\int_{-\infty}^{x}e^y\frac{1}{\sqrt{2\pi}\sigma_0}\exp\{-\frac{(y-\mu_0)^2}{2\sigma_0{}^2}\}dy\\&=\frac{1}{\sqrt{2\pi}\sigma_0}\int_{-\infty}^{x}\exp\{-\frac{(y-(\mu_0+\sigma_0{}^2))^2}{2\sigma_0{}^2}\}dy\end{aligned}$$

III. MAIN RESULTS

Before we want the formula for the value of a European-type option. This option is simultaneously a call option on asset S_t^2 with exercise price S_t^1 and a put option on asset S_t^1 with exercise price S_t^2.

Proposition 1 Assume that the dynamics of a bond B_t and two risky assets S_t^1, S_t^2 are given by (1),(2),(3), respectively. Then the price of a European-type option with a call option on asset S_t^2 with exercise price S_t^1 and expiry date T is given by the expression

$$\begin{aligned}C(0,S_T^1,S_T^2)=\sum_{n=0}^{\infty}P_n(T)E[&S_0^2\exp\{(1-E(U))\int_0^T\lambda(s)ds\}\prod_{i=1}^{n}U_i\Phi(d_1)-\\&S_0^1\exp\{(1-E(V))\int_0^T\lambda(s)ds\}\prod_{i=1}^{n}V_i\Phi(d_2)]\end{aligned}$$

where $d_1=\dfrac{\ln\dfrac{S_0^2\prod_{i=1}^{n}\frac{U_i}{V_i}}{S_0^1}+(E(V)-E(U))\int_0^T\lambda(s)ds+\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}$,

$d_2=d_1-\sigma\sqrt{T}$, $\sigma=\sqrt{\sigma_1{}^2-2\sigma_1\sigma_2\rho+\sigma_2{}^2}$.

Proof Since P is risk-neutral martingale measure,we have

$$C(0,S_T^1,S_T^2)=E[\frac{1}{B_T}(S_T^2-S_T^1)^+]\ =E\left[\frac{S_T^2}{B_T}I_{\{S_T^2\ge S_T^1\}}\right]-E\left[\frac{S_T^1}{B_T}I_{\{S_T^2\ge S_T^1\}}\right]\tag{4}$$

For $E\left[\frac{S_T^2}{B_T}I_{\{S_T^2\ge S_T^1\}}\right]$, let $X_t=\frac{S_t^1}{S_t^2}$, applying Ito's lemma yield[9]

$$\frac{dX_t}{X_{t^-}}=(\sigma_2{}^2-\sigma_1\sigma_2\rho)dt+(\sigma_1\rho-\sigma_2)dW_t^2+\sigma_1\sqrt{1-\rho^2}dW_t^3+\frac{V-U}{U}d\tilde{N}_t\tag{5}$$

or equivalently

$$\frac{dX_t}{X_{t^-}}=\sigma dW_t+\frac{V-U}{U}d\tilde{N}_t\tag{6}$$

where $\sigma=\sqrt{\sigma_1{}^2-2\sigma_1\sigma_2\rho+\sigma_2{}^2}$, $W_t=\frac{\sigma_1\rho-\sigma_2}{\sigma}(W_t^2-\sigma_2t)+\frac{\sigma_1\sqrt{1-\rho^2}}{\sigma}W_t^3$.

Definite by

$$\frac{dP^*}{dP}=\frac{S_T^2}{S_0^2B_T}=\prod_{i=1}^{N_T}U_i\exp\{-\frac{1}{2}\sigma_2{}^2T+\sigma_2W_T^2+(1-E(U))\int_0^T\lambda(s)ds\}$$

It is not hard to check that W_t^3 and $W_t^2-\sigma_2t(0\le t\le T)$ follow one-dimensional standard Wiener processes under the martingale measure P^*, and their mutually independent. Then it is easy that $W_t=\frac{\sigma_1\rho-\sigma_2}{\sigma}(W_t^2-\sigma_2 t)+\frac{\sigma_1\sqrt{1-\rho^2}}{\sigma}W_t^3$ is standard one-dimensional Wiener process under the martingale measure P^*.

The unique solution of stochastic differential equation (6)equals[10]

$$X_T=X_0\prod_{i=1}^{N_T}(\frac{V_i}{U_i})\exp\{-\frac{1}{2}\sigma^2T+\sigma W_T-E_*[\frac{V-U}{U}]E(U)\int_0^T\lambda(s)ds\} \tag{7}$$

By Lemma 1 $E_*[\frac{V-U}{U}]=E[\frac{V-U}{U}\cdot\frac{U}{E(U)}]=\frac{E(V)-E(U)}{E(U)}$

(7) equivalently

$$X_T=X_0\prod_{i=1}^{N_T}(\frac{V_i}{U_i})\exp\{-\frac{1}{2}\sigma^2T+\sigma W_T+(E(U)-E(V))\int_0^T\lambda(s)ds\} \tag{8}$$

let $X_T^n=X_0\prod_{i=1}^{n}(\frac{V_i}{U_i})\exp\{-\frac{1}{2}\sigma^2T+\sigma W_T+(E(U)-E(V))\int_0^T\lambda(s)ds\}$

thus

$$\begin{aligned}E\left[\frac{S_T^2}{B_T}I_{\{S_T^2\ge S_T^1\}}\right]&=S_0^2E_*[I_{\{X_T\le 1\}}]=S_0^2\sum_{n=0}^{\infty}P_n^*(T)E_*[P^*(X_T^n\le 1)]\\&=S_0^2\sum_{n=0}^{\infty}P_n(T)(E(U))^n\exp\{(1-E(U))\int_0^T\lambda(s)ds\}E_*[P^*(X_T^n\le 1)]\\&=S_0^2\sum_{n=0}^{\infty}P_n(T)\exp\{(1-E(U))\int_0^T\lambda(s)ds\}E[P^*(X_T^n\le 1)\prod_{i=1}^{n}U_i]\end{aligned} \tag{9}$$

$$\begin{aligned}&P^*(X_T^n\le 1)=P^*(\ln X_T^n\le 0)\\&=P^*(\ln X_0\prod_{i=1}^{n}(\frac{V_i}{U_i})\exp\{-\frac{1}{2}\sigma^2T+\sigma W_T+(E(U)-E(V))\int_0^T\lambda(s)ds\}\le 0)\\&=P^*(\sigma W_T\le\ln\frac{S_0^2\prod_{i=1}^{n}\frac{U_i}{V_i}}{S_0^1}+(E(V)-E(U))\int_0^T\lambda(s)ds+\frac{1}{2}\sigma^2T)\\&=P^*(\sigma W_T\le d)\\&=\Phi(\frac{d}{\sigma\sqrt{T}})\end{aligned} \tag{10}$$

where $d=\ln\frac{S_0^2\prod_{i=1}^{n}\frac{U_i}{V_i}}{S_0^1}+(E(V)-E(U))\int_0^T\lambda(s)ds+\frac{1}{2}\sigma^2T$

By virtue of (9)and(10),we have

$$E\left[\frac{S_T^2}{B_T}I_{\{S_T^2\ge S_T^1\}}\right]=S_0^2\sum_{n=0}^{\infty}P_n(T)\exp\{(1-E(U))\int_0^T\lambda(s)ds\}E[\Phi(d_1)\prod_{i=1}^{n}U_i] \tag{11}$$

where

$$d_1=\frac{d}{\sigma\sqrt{T}}=\frac{\ln\frac{S_0^2\prod_{i=1}^{n}\frac{U_i}{V_i}}{S_0^1}+(E(V)-E(U))\int_0^T\lambda(s)ds+\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}$$

For $E\left[\frac{S_T^1}{B_T}I_{\{S_T^2\ge S_T^1\}}\right]$, let $Y_t=\frac{S_t^2}{S_t^1}$,

$$\frac{dP^*}{dP}=\frac{S_T^1}{S_0^1B_T}=\prod_{i=1}^{N_T}V_i\exp\{-\frac{1}{2}\sigma_1^2T+\sigma_1W_T^1+(1-E(V))\int_0^T\lambda(s)ds\}$$

By using the same way we have

$$E\left[\frac{S_T^1}{B_T}I_{\{S_T^2\ge S_T^1\}}\right]=S_0^1\sum_{n=0}^{\infty}P_n(T)\exp\{(1-E(V))\int_0^T\lambda(s)ds\}E[\Phi(d_2)\prod_{i=1}^{n}V_i] \tag{12}$$

Where

$$d_2=\frac{d-\sigma^2T}{\sigma\sqrt{T}}=\frac{\ln\frac{S_0^2\prod_{i=1}^{n}\frac{U_i}{V_i}}{S_0^1}+(E(V)-E(U))\int_0^T\lambda(s)ds-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}=d_1-\sigma\sqrt{T}$$

Together with(4), (11)and(12),we have proved the following

$$\begin{aligned}C(0,S_T^1,S_T^2)=\sum_{n=0}^{\infty}P_n(T)E[&S_0^2\exp\{(1-E(U))\int_0^T\lambda(s)ds\}\prod_{i=1}^{n}U_i\Phi(d_1)-\\&S_0^1\exp\{(1-E(V))\int_0^T\lambda(s)ds\}\prod_{i=1}^{n}V_i\Phi(d_2)]\end{aligned}$$

Proposition 2 Assume that the dynamics of a bond B_t and two risky assets S_t^1, S_t^2 are given by (1),(2),(3), respectively. If V,U have lognormal distribution[11] with mean parameter μ_V,μ_U and variance σ_V^2,σ_U^2 under the martingale measure P,then the price of a European-type option with a call option on asset S_t^2 with exercise price S_t^1 and expiry date T is given by the expression

$$\begin{aligned}C(0,S_T^1,S_T^2)=\sum_{n=0}^{\infty}[&P_n(T)S_0^2\mu_U^n\exp\{(1-\mu_U)\int_0^T\lambda(s)ds\}\Phi(a_1)-\\&S_0^1\mu_V^n\exp\{(1-\mu_V)\int_0^T\lambda(s)ds\}\Phi(a_2)]\end{aligned}$$

where $a_1=\frac{\ln\frac{S_0^2}{S_0^1}+(\mu_V-\mu_U)\int_0^T\lambda(s)ds+\frac{1}{2}\sigma^2T-n(\mu_V-\mu_U-\sigma_U^2)}{\sqrt{\sigma^2T+n(\sigma_V^2+\sigma_U^2)}}$,

$a_2=a_1-\sqrt{\sigma^2T+\sigma_V^2+\sigma_U^2}$, $\sigma=\sqrt{\sigma_1^2-2\sigma_1\sigma_2\rho+\sigma_2^2}$.

Proof Using **Proposition 1**

$$C(0,S_T^1,S_T^2)=\sum_{n=0}^{\infty}P_n(T)\{S_0^2\exp\{(1-\mu_U)\int_0^T\lambda(s)ds\}E[\prod_{i=1}^{n}U_i\Phi(d_1)]-S_0^1\exp\{(1-\mu_V)\int_0^T\lambda(s)ds\}E[\prod_{i=1}^{n}V_i\Phi(d_2)]\}$$

By using Lemma 1, we have $\sigma W_t-\sum_{i=1}^{n}\ln U_i+\sum_{i=1}^{n}\ln V_i$ would be a normal random variable under the martingale measure P^* with mean and variance given by

$$E_*[\sigma W_t-\sum_{i=1}^{n}\ln U_i+\sum_{i=1}^{n}\ln V_i]=n(\mu_V-\mu_U-\sigma_U{}^2),$$

$$D_*[\sigma W_t-\sum_{i=1}^{n}\ln U_i+\sum_{i=1}^{n}\ln V_i]=\sigma^2t+n(\sigma_V{}^2+\sigma_U{}^2)$$

then $E[\prod_{i=1}^{n}U_i\Phi(d_1)]=\mu_U{}^nE_*[\Phi(d_1)]$

$$=\mu_U{}^nP^*(\sigma W_T\le\ln\frac{S_0^2\prod_{i=1}^{n}\frac{U_i}{V_i}}{S_0^1}+(E(V)-E(U))\int_0^T\lambda(s)ds+\frac{1}{2}\sigma^2T)$$

$$=\mu_U{}^nP^*(\sigma W_T-\sum_{i=1}^{n}\ln U_i+\sum_{i=1}^{n}\ln V_i\le a)$$

$$=\mu_U{}^n\Phi(\frac{a-n(\mu_V-\mu_U-\sigma_U{}^2)}{\sqrt{\sigma^2T+n(\sigma_V{}^2+\sigma_U{}^2)}})$$

$$=\mu_U{}^n\Phi(a_1)$$

and $E[\prod_{i=1}^{n}V_i\Phi(d_2)]=\mu_V{}^n\Phi(a_2)$

where $a=\ln\frac{S_0^2}{S_0^1}+(\mu_V-\mu_U)\int_0^T\lambda(s)ds+\frac{1}{2}\sigma^2T$

Hence, we have

$$C(0,S_T^1,S_T^2)=\sum_{n=0}^{\infty}[P_n(T)S_0^2\mu_U{}^n\exp\{(1-\mu_U)\int_0^T\lambda(s)ds\}\Phi(a_1)-S_0^1\mu_V{}^n\exp\{(1-\mu_V)\int_0^T\lambda(s)ds\}\Phi(a_2)]$$

Proposition 3 Assume that the dynamics of a bond B_t and two risky assets S_t^1, S_t^2 are given by (1),(2),(3), respectively. Then the put-call parity relation may be rewritten as

$$C\left(t,S_T^1,S_T^2\right)-P\left(t,S_T^1,S_T^2\right)=S_t^2-S_t^1.$$

We can using put-call parity[12] to find the price of a European put option on a stock with the same parameters as earlier.

IV. Concluding Remarks

In this paper, we develop an equation for the value of the option to exchange one risky asset for another. Establish the option-pricing model when exercise price is random variable. The option-pricing model is options to exchange one asset to another. Because jumps do occur in practice, it is advantageous to consider a model for price evolution that superimposes random jumps on a geometric Brownian motion. By changing basic assumption of William Margrabe exchange option pricing model to the assumption that jump process is count process that more general than Poisson process, it is established that the behavior model of the stock pricing process is jump-diffusion process. Considering when the jump distribution is lognormal, we get European exchange call and put option pricing formula and their parity.

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