Superconvergence of Variational Discretization for Bilinear Elliptic Optimization Problems

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Abstract—**In this paper, variational discretization approximation of bilinear elliptic optimal control problems is considered. Firstly, we construct a variational discretization method for the bilinear elliptic optimal control problem with control constraints. Secondly, we derive the convergence of the approximation scheme. Thirdly, we analyze the superconvergence. Finally, we present a numerical example to conform our theoretical results.**

Keywords-convergence; superconvergence; variational discretization; bilinear elliptic optimal control problems

I. INTRODUCTION

There has been so extensive research on convergence and super convergence analysis of finite element method for elliptic optimization that it is impossible to given a very brief review here. Convergence and super convergence for finite element approximation of linear or semi-linear elliptic optimization problems are investigated in [1, 2] and [3, 4], respectively.

In recent years, much research has begun to examine finite element approximations of bilinear optimal control problems. A priori error estimates of finite element discretization for bilinear elliptic optimal control problems is established in [5]. Super convergence of finite element method for optimal control problems of bilinear type is obtained in [6].

Hinze present a vatiational discretization (VD) concept for control constrained optimization problems in [7]. Its main feature is not directly to discretize the space of admissible controls but to obtain the control by utilizing the discretization of the state and co-state variables and the implicity solution of variational inequality. It can improve the convergent order from $O(h)$ to $O(h^2)$. Recent years, VD are used to solve different kinds of constrained. In this paper, we consider the VD approximation of bilinear elliptic optimization problems with control constraints.

We are interested in the following optimization problem:

$$\min_{u \in K} \frac{1}{2} \int_\Omega (|y(x) - y_d(x)|^2 + |u(x)|^2)dx \tag{1}$$

subject to

$$-div(A(x)\nabla y(x) + u(x)y(x) = f(x), \quad x \in \Omega, \tag{2}$$

$$y(x) = 0, \quad x \in \partial\Omega, \tag{3}$$

where the set of admissible control is defined by

$$K = \left\{ v(x) \in L^2(\Omega) : a \le v(x) \le b \right\} \tag{4}$$

and $0 < a < b < +\infty$ are constants.

Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a bounded open domain with a smooth boundary $\partial\Omega$. The coefficient A is a symmetric matrix such that $\xi^T A\xi \ge c \| \xi \|^2, \xi \in \mathbb{R}^d$ with $c > 0$. In this paper, we use the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with standard norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$.

II. VD APPROXIMATION FOR THE MODEL PROBLEM

In this section, we present a VD approximation for the optimization problem (1.1)-(1.4). For ease of exposition, we set $W_0^{m,q}(\Omega) \equiv \left\{ v \in W^{m,q}(\Omega) : v|_{\partial\Omega} = 0 \right\}, H^1(\Omega) = W^{m,2}(\Omega)$ and $H_0^1(\Omega) = W_0^{m,2}(\Omega)$. Furthermore, we set $U = L^2(\Omega)$ and $W = H_0^1(\Omega)$ and denote $\|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|_m$ and $\|\cdot\|$, respectively.

Let

$$a(v, w) = \int_\Omega (A\nabla v) \cdot \nabla w, \quad \forall v, w \in W,$$

$$(f_1, f_2) = \int_\Omega f_1 \cdot f_2, \quad \forall f_1, f_2 \in U.$$

It follows from the assumption on A, we have that

$$a(v, v) \ge c \| v \|_1^2, \quad \forall v \in W,$$

$$a(v, w) \le C \| v \|_1 \| w \|_1, \quad \forall v, w \in W.$$

Then a weak formula for the model problem reads:

$$\begin{cases} \min_{u \in K} \frac{1}{2} \left\{ \| y - y_d \|^2 + \| u \|^2 \right\} \\ a(y,w) + (uy,w) = (f,w), \qquad \forall w \in W. \end{cases} \quad (5)$$

It is well known (see, e.g., [8]) that the control problem (5) has a solution (y,u), and that a pair (y,u) is the solution of (5), then there is a co-state $p \in W$ such that the triplet (y,p,u) satisfies the following optimality conditions:

$$a(y,w) + (uy,w) = (f,w), \qquad \forall w \in W, \quad (6)$$

$$a(q,p) + (up,q) = (y - y_d, q), \qquad \forall q \in W, \quad (7)$$

$$(u - yp, v - u) \geq 0, \qquad \forall v \in K. \quad (8)$$

As in [9], it is easy to prove the following lemma:

Lemma 2.1 Let (y,p,u) be the solution of (6)-(8). Then

$$u = \min(\max(a, yp), b). \quad (9)$$

Let T^h and h be regular triangulations of Ω and the mesh size, respectively. Moreover, we set

$$W_h = \left\{ v_h \in C(\overline{\Omega}) : v_h \mid_\tau \in P_1, \forall \tau \in T^h, v_h \mid_{\partial\Omega} = 0 \right\},$$

Where P_1 denotes polynomials of total degree no more than 1. Thus a VD approximation of (6)-(8) is as follows:

$$\begin{cases} \min_{u_h \in K} \frac{1}{2} \left\{ \| y - y_d \|^2 + \| u_h \|^2 \right\} \\ a(y_h, w_h) + (u_h y_h, w_h) = (f, w_h), \quad \forall w_h \in W_h. \end{cases} \quad (10)$$

The problem (10) again has a solution (y_h, u_h) and that if $(y_h, u_h) \in W_h \times K$ is the solution of (10), then there is a co-state $p_h \in W_h$, such that the triplet (y_h, p_h, u_h) satisfies the following optimality conditions:

$$a(y_h, w_h) + (u_h y_h, w_h) = (f, w_h), \quad \forall w_h \in W_h, \quad (11)$$

$$a(q_h, p_h) + (u_h p_h, q_h) = (y_h - y_d, q_h), \quad \forall q_h \in W_h, \quad (12)$$

$$(u_h - y_h p_h, v - u_h) \geq 0, \qquad \forall v \in K. \quad (13)$$

Similar to the Lemma 2.1, we can easily derive the following lemma:

Lemma 2.2 Let (y_h, p_h, u_h) be the solution of (11)-(13). Then, we have

$$u_h = \min(\max(a, y_h p_h), b). \quad (14)$$

Remark 2.1 We minimize over the infinite dimensional set K instead of minimizing over a finite dimensional subset of K in (13). Then we just need to solve the discrete equations (11)-(12) and obtain u_h from (14).

III. ERROR ESTIMATES OF INTERMEDIATE VARIABLES

Some intermediate variables and useful error estimation are introduced in this section. For any $v \in K, w_h, q_h \in W_h$, let $y_h(v), p_h(v) \in W_h$ satisfies the following system:

$$a(y_h(v), w_h) + (vy_h(v), w_h) = (f, w_h), \quad (15)$$

$$a(q_h, p_h(v)) + (vp_h(v), q_h) = (y_h(v) - y_d, q_h). \quad (16)$$

If (y_h, p_h, u_h) be the solution of (11)-(13) then we have $(y_h, p_h) = (y_h(u_h), p_h(u_h))$.

The elliptic projection operator $R_h : W \to W_h$ defined as follows: for any $\phi \in W$ satisfies

$$a(\phi - R_h \phi, w_h) = 0, \qquad \forall w_h \in W_h.$$

It has the following approximation property (see e.g., [10])

$$\| \varphi - R_h \varphi \|_s \leq Ch^{2-s} \| \varphi \|_2, \quad \forall \varphi \in H^2(\Omega), s = 0,1. \quad (17)$$

Lemma 3.1 Let (y, p, u) be the solution of (6)-(8) and $(y_h(u), p_h(u))$ be the solution of (15)-(16) with $v = u$. Suppose $u \in L^\infty(\Omega)$ and $y, p \in H^2(\Omega)$, then we have

$$\| y_h(u) - y \| + \| p_h(u) - p \| \leq Ch^2. \quad (18)$$

Prof. Set $v = u$ in (15), then from (6) and the definition of the elliptic projection operator R_h, we have

$$a(y_h(u) - R_h y, w_h) + (u(y_h(u) - R_h y), w_h)$$
$$= (u(y - R_h y), w_h), \qquad \forall w_h \in W_h. \tag{19}$$

Note that $u \in K \cap L^\infty(\Omega)$ and

$$a(y_h(u) - R_h y, y_h(u) - R_h y) \geq c \parallel y_h(u) - R_h y \parallel. \tag{20}$$

From (17) and (19)-(20), we obtain

$$c \parallel y_h(u) - R_h y \parallel \leq Ch^2 \parallel y \parallel_2 . \tag{21}$$

Hence, from (17) and (21), we derive

$$\parallel y_h(u) - y \parallel \leq \parallel y_h(u) - R_h y \parallel + \parallel R_h y - y \parallel$$
$$\leq Ch^2 \parallel y \parallel_2 . \tag{22}$$

By choosing $v = u$ in (16), then from (7) and the definition of the elliptic projection operator, we get

$$a(q_h, p_h(u) - R_h p) + (u(p_h(u) - R_h p), q_h)$$
$$= (u(p - R_h p), q_h) + (y_h(u) - y, q_h), \ \forall w_h \in W_h. \tag{23}$$

Similarly, we can derive

$$c \parallel p_h(u) - p \parallel \leq Ch^2 (\parallel p \parallel_2 + \parallel y \parallel_2). \tag{24}$$

Then (18) follows from (22) and (24).

IV.　CONVERGENCE ANALYSIS

In this section, we will give the convergence analysis for the control variable. Just for ease of exposition, we set

$$J(u) = \frac{1}{2} \left\{ \parallel y - y_d \parallel^2 + \parallel u \parallel^2 \right\},$$
$$J_h(u_h) = \frac{1}{2} \left\{ \parallel y_h - y_d \parallel^2 + \parallel u_h \parallel^2 \right\}.$$

It can be shown that

$$J'(u) = (u - yp, u), \tag{25}$$

$$J'_h(u_h) = (u_h - y_h(u_h) p_h(u_h), u_h). \tag{26}$$

In many applications, the objective functional $J(\cdot)$ is uniform convex near the solution u (see e.g., [11]). As in [12], we assume that there is a positive constant C, such that

$$C \parallel u - v \parallel^2 \leq (J'_h(u) - J'_h(v), u - v), \ \forall u, v \in K. \tag{27}$$

Theorem 4.1 Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (6)-(8) and (11)-(13), respectively. Assume that $y_h(u), p \in L^\infty(\Omega)$ and all the conditions in the Lemma 3.1 are valid. Then, we have

$$\parallel u - u_h \parallel \leq Ch^2. \tag{28}$$

Prof. Set $v = u_h$ and $v = u$ in (8) and (13), respectively. We obtain

$$(u - yp, u_h - u) \geq 0. \tag{29}$$

and

$$(u_h - y_h p_h, u - u_h) \geq 0. \tag{30}$$

From (25)-(27) and (29)-(30), we have

$$C \parallel u - u_h \parallel^2$$
$$\leq (J'_h(u) - J'_h(u_h), u - u_h)$$
$$= (u - y_h(u) p_h(u) - u_h + y_h(u_h) p_h(u_h), u - u_h)$$
$$\leq ((y - y_h(u))p + y_h(u)(p - p_h(u)), u - u_h)$$
$$\leq C(\varepsilon)(\parallel y - y_h(u) \parallel^2 + \parallel p - p_h(u) \parallel^2)$$
$$+ 2\varepsilon \parallel u - u_h \parallel^2 . \tag{31}$$

Let ε be small enough, then (28) follows from (18) and (31).

V.　SUPER CONVERGENCE ANALYSIS

In this section, we will analyze the super convergence of the state and co-state variables.

Theorem 5.1 Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (6)-(8) and (11)-(13), respectively. Suppose all the conditions in the Theorem 4.1 hold. We have

$$\| y_h - R_h y \|_1 + \| p_h - R_h p \|_1 \le Ch^2. \tag{32}$$

Prof. From (6) and (11), for any $w_h \in W_h$, we get

$$a(y - y_h, w_h) + (u(y - y_h), w_h)$$
$$= (y_h(u_h - u), w_h). \tag{33}$$

According to the definition of R_h, we obtain

$$a(R_h y - y_h, w_h) + (u(R_h y - y_h), w_h)$$
$$= (u(R_h y - y_h), w_h) + (y_h(u_h - u), w_h) \tag{34}$$

Selecting $w_h = R_h y - y_h$, (17), (28) and (34), we derive

$$\| R_h y - y_h \|_1 \le Ch^2. \tag{35}$$

From (7) and (12), for any $q_h \in W_h$, we have

$$a(q_h, p - p_h) + (u(p - p_h), q_h)$$
$$= (p_h(u_h - u), q_h) + (y - y_h, q_h). \tag{36}$$

Similarly, we can derive

$$\| R_h p - p_h \|_1 \le Ch^2. \tag{37}$$

Hence, (32) follows from (35) and (37).

VI. NUMERICAL EXPERIMENT

Let $\Omega = [0,1] \times [0,1], c = 0, b = 1$ and A is an unit matrix.

We solve the following optimal control problem where codes developed based on AFEPack which is freely available.

Example. The data are as follows:

$$y(x) = \sin(2\pi x_1)\sin(2\pi x_2)$$
$$p(x) = y(x)$$
$$u(x) = \min(\max(-0.5, y(x)p(x)), 0.5)$$
$$f(x) = -div(A\nabla y(x)) + u(x)y(x)$$
$$y_d(x) = y(x) + div(A\nabla p(x)) - u(x)p(x)$$

The errors $\| u - u_h \|, \| y_h - R_h y \|_1, \| p_h - R_h p \|_1$ based on a sequence of uniform refined meshes are shown in table I.

TABLE I. THE ERRORS ON UNIFORM MESH

Mesh nodes	Errors		
	$\| u - u_h \|$	$\| y_h - R_h y \|_1$	$\| p_h - R_h p \|_1$
100	4.97429e-02	8.36251e-03	7.42735e-03
400	1.31232e-02	2.30218e-03	2.00054e-03
1600	3.31625e-03	5.91429e-04	5.10980e-04
6400	8.31263e-04	1.48953e-04	1.28457e-04

The superconvergence phenomenon can be observed clearly from the Table I.

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