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Several Forms of Li-Fan Integrable Unified Equation

Xingju Dang[1], Yuhui Luo[2] and Wenliang Wu[3,*]
[1,2,3]College of Physics & Electronic Information Engineering, Zhaotong University, Zhaotong Yunnan, 657000, China
[*]Corresponding author

***Abstract*—The naming the equation $\mathrm{d}y/\mathrm{d}x+\{[(\mathrm{d}g/\mathrm{d}x)w+\mathrm{d}\psi/\mathrm{d}x]/g(x)-q(x)F[g(x)w(y)+\psi(x)]\}/(\mathrm{d}w/\mathrm{d}y)=0$ for Li-Fan Integrable Unified Equation is suggested. Several forms of the equation are given.**

Keywords-ordinary differential equation; nonlinear equation; first order equation; general integral; integrable unified equation

I. INTRODUCTION

In 1982, Prof. Hongxiang Li gave a first order differential integrable equation in [1]:

$$y' + P(x)y = y^n Q(x) F\left(y e^{\int P(x)\mathrm{d}x} \right). \tag{1}$$

He pointed out that separable variable equation, linear equation, homogeneous equation and Bernoulli Equation are special cases of (1). In 1999, Prof. Li referred to (1) as unified equation of these integration equations in [2].

In 1987, Prof. Li and Xing Fan expanded (1) into the following form (Theorem 1.2 in [3]):

$$\begin{gathered} f(x)w'(y)y' + \left[P(x)f(x) + f'(x)\right]w(y) = \\ \left[f(x)w(y)+\varphi(x)\right]^{\alpha} Q(x)F_1\left(\left[f(x)w(y)+\varphi(x)\right]v(x)\right) - \\ P(x)\varphi(x) - \varphi'(x), \end{gathered} \tag{2}$$

where

$$v' - P(x)v = kQ(x)v^{2-\alpha}. \tag{3}$$

Its general integral is

$$\int \frac{\mathrm{d}u}{u^{\alpha}F_1(u) + ku} = \int Q(x)v^{1-\alpha}(x)f(x)\mathrm{d}x + A. \tag{4}$$

where A is a arbitrary constant,

$$u = v(x)\left[f(x)w(y) + \varphi(x)\right]. \tag{5}$$

In 1990, Prof. Li and Z.F.Starc discussed a class of more general first order ordinary differential integration equations, and gave some theorems and their corollaries in [4]. Equation (2) is mentioned as a corollary.

In order to facilitate the application, this paper simplifies (2), and give its several forms.

II. THEOREM

Suppose that $q, F \in \mathrm{C}$; $g, w, \psi \in \mathrm{C}^1$; $g(x)\neq 0$ and $w(y)\neq$const, Equation

$$y' + \left(\frac{\mathrm{d}w}{\mathrm{d}y}\right)^{-1} \left\{ \frac{g'(x)w(y)+\psi'(x)}{g(x)} - q(x)F\left[gw+\psi\right] \right\} = 0 \tag{6}$$

can be integrated. Its general integral is

$$\int \frac{\mathrm{d}u}{F(u)} = \int g(x)q(x)\mathrm{d}x + C. \tag{7}$$

where

$$u = g(x)w(y) + \psi(x). \tag{8}$$

C is a arbitrary constant.

III. PROVE

Let $u = g(x)w(y) + \psi(x)$, then

$$\begin{gathered} u' = g'(x)w(y) + g(x)w'(y)y' + \psi'(x) = \\ g(x)\left\{ q(x)F\left[g(x)w(y)+\psi(x)\right] - \frac{\psi'(x)}{g(x)} \right\} + \psi'(x) = \\ g(x)q(x)F\left[g(x)w(y)+\psi(x)\right] = g(x)q(x)F(u). \end{gathered}$$

It is a separable equation and its general solution is (7). Then the theorem is proved.

Let $g(x) = v(x)f(x)$, $\psi(x) = v(x)\varphi(x)$,

$q(x) = \left[v(x)\right]^{\alpha} Q(x)$, $F(u) = u^{\alpha}F_1(u) + ku$,

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if the requirement (3) was founded, then (6) becomes (2), and (7) becomes (4).

IV. COROLLARIES

Corollary 1 Suppose that P, q, $F \in \mathrm{C}$; w, $\psi \in \mathrm{C}^1$; $w(y) \neq$ const. Equation

$$y' + \frac{P(x)w(y) + \psi'(x)\mathrm{e}^{-\int P(x)\mathrm{d}x} - q(x)F\left[w\mathrm{e}^{\int P(x)\mathrm{d}x} + \psi\right]}{\mathrm{d}w/\mathrm{d}y} = 0 \qquad (9)$$

is integration.

Proof In (6), taking $g(x)=\mathrm{e}^{\int P(x)\mathrm{d}x}$, (6) becomes (9). According to Theorem the proof is ended.

Corollary 2 Suppose that f, q, $F \in \mathrm{C}$; g, $\psi \in \mathrm{C}^1$; $g(x) \neq 0$. Equation

$$y' = \frac{g'(x)}{g(x)}f(y) + f(y)\mathrm{e}^{\int \frac{\mathrm{d}y}{f(y)}}\left\{\frac{\psi'(x)}{g(x)} - q(x)F\left[g(x)\mathrm{e}^{-\int \frac{\mathrm{d}y}{f(y)}} + \psi\right]\right\} \qquad (10)$$

is integration.

Proof In (6), taking $w(y)=\mathrm{e}^{-\int \frac{1}{f(y)}\mathrm{d}y}$, (6) becomes (10). According to Theorem the proof is ended.

Corollary 3 Suppose that P, q, f, $F \in \mathrm{C}$; $\psi \in \mathrm{C}^1$, $f(y) \neq 0$. Equation

$$y' = P(x)f(y) + f\exp\left(\int \mathrm{d}y/f\right)\left\{\psi'(x)\exp\left(-\int P\mathrm{d}x\right) - q(x)F\left[\exp\left(\int P\mathrm{d}x - \int \mathrm{d}y/f\right) + \psi\right]\right\} \qquad (11)$$

Is integrable.

Proof In (6), taking $w(y)=\mathrm{e}^{-\int \frac{1}{f(y)}\mathrm{d}y}$, $g(x)=\mathrm{e}^{\int P(x)\mathrm{d}x}$, (6) becomes (11). According to Theorem the proof is ended.

We suggest the naming (5), (9),(10) and (11) for Li-Fan Integrable Unified Equations when the requirements in the theorem or corollarys was founded.

Corollary 4 Suppose that P, q, $F \in \mathrm{C}$, $\psi \in \mathrm{C}^1$. Equation

$$y' = P(x) \pm \mathrm{e}^{\pm y}\left\{\psi'(x)\mathrm{e}^{-\int P\mathrm{d}x} - q(x)F\left[\mathrm{e}^{\int P\mathrm{d}x \mp y} + \psi\right]\right\} \qquad (12)$$

is integrable.

Proof In (11), let $f(y) \equiv \pm 1$, if $f(y) \equiv -1$, then $P(x)$ is replaced by $-P(x)$, (11) becomes (12). According to Corollary 3 the proof is ended.

Corollary 5 Suppose that P, q, $F \in \mathrm{C}$; $\psi \in \mathrm{C}^1$. Equation

$$y' = P(x)y + y^2\left\{\psi'(x)\mathrm{e}^{-\int P\mathrm{d}x} - q(x)F\left[\frac{\mathrm{e}^{\int P\mathrm{d}x}}{y} + \psi\right]\right\} \qquad (13)$$

is integration.

Proof In (11), let $f(y)=y$, (11) becomes (13). According to Corollary 3 the proof is ended.

Corollary 6 Suppose that P, q, $F \in \mathrm{C}$, $\psi \in \mathrm{C}^1$. Equation

$$y' = -P(x)y + \left\{\psi'(x)\mathrm{e}^{-\int P\mathrm{d}x} - q(x)F\left[y\mathrm{e}^{\int P\mathrm{d}x} + \psi\right]\right\} \qquad (14)$$

is integration.

Proof In (11), let $f(y)=-y$, (11) becomes (14). According to Corollary 3 the proof is ended.

In (14), let $\psi(x) \equiv 0$, $q(x) = Q(x)\exp\left(n\int P(x)\,\mathrm{d}x\right)$, $F(u)$ is replaced by $F(u)u^{-n}$, (14) becomes (1).

Corollary 7 Suppose that q, $\Phi \in \mathrm{C}$; m, $n \in \mathrm{C}^1$. Equation

$$y' = -\frac{m'(x)y + n'(x)}{m(x)} + q(x)\Phi\left[m(x)y + n(x)\right] \qquad (15)$$

is integration.

Proof In (6), let $w(y)=y$, $g(x)=m(x)$, $\psi(x)=n(x)$, $F(u)=\Phi(u)$, (6) becomes (15). According to Theorem the proof is ended. In fact, this corollary is Theorem in [5] by Prof. Luxiang Feng in 2013.

Corollary 8 If equation

$$y' - a\lambda(x)y = q(x)\Phi\left[(by + c)G(\lambda)\right] + \frac{ac}{b}\lambda \qquad (16)$$

meet requirement $\left[\frac{1}{G(\lambda)}\right]' - \frac{a\lambda}{G} = kq$, then it is integration.

Proof In (2), let $w(y)=by+c$, $g(x) \equiv 1$, $\varphi(x) \equiv 0$, $v(x)=G(\lambda(x))$, $F_1(u)=\Phi(u)$, $P(x)=-a\lambda(x)$, $\alpha=0$, (3) is met. (2) becomes (16). According to the prove of Theorem the proof is ended. In fact, this corollary is Theorem in [6] by Prof. Feng in 2012 (d is replaced by k here).

Corollary 9 Equation

$$y' - \frac{h'}{h}y = q\Phi\left[\frac{y+f}{h}\right] + \frac{h'}{h}f - f' + \alpha q \qquad (17)$$

is integration.

Proof In (6), let $w(y)=y$, $g(x)=1/h(x)$, $\psi(x)=f(x)/h(x)$, $F(u)=\Phi(u)+\alpha$, then

$$\frac{g'}{g}=h\left(\frac{1}{h}\right)'=-\frac{h'}{h}.$$

(6) becomes (17). According to Theorem the proof is ended. In fact, this corollary is Theorem in [7] by Prof. Zhilin Li in 2009 (g is replaced by h here).

Corollary 10 Equation

$$y'=p(x)y+Q(x)(y+f(x))^n+p(x)f(x)-f'(x) \tag{18}$$

is integration.

Proof In (9), let $w(y)=y$, $P(x)=-p(x)$, $g(x)=\exp\left(-\int p(x)\,\mathrm{d}\,x\right)$, $q(x)=Q(x)\exp\left(n\int p(x)\,\mathrm{d}\,x\right)$, $\psi(x)=f(x)\exp\left(-\int p(x)\,\mathrm{d}\,x\right)$, $F(u)=u^n$, then

$$\psi'(x)\exp\left(-\int P(x)\mathrm{d}x\right)=\psi'(x)\exp\left(\int p(x)\mathrm{d}x\right)=$$
$$\left[\psi(x)\exp\left(\int p(x)\mathrm{d}x\right)\right]'-\psi(x)\exp\left(\int p(x)\mathrm{d}x\right)p(x)=$$
$$f'(x)-p(x)f(x).$$

(9) becomes (18). According to Corollary 1 the proof is ended. In fact, this corollary is Theorem 1 in [8] (q is replaced by Q here).

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